CHAPTER 2. FIRST ORDER DIFFERENTIAL EQUATIONS

2.4. Exact Differential Equations

Definition. Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy$$

for all $(x,y) \in D$.

Definition The differential form

$$M(x,y)dx + N(x,y)dy \tag{1}$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in D$.

That is, expression (1) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y) \text{ for all } (x,y) \in D.$$

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an exact differential equation.

Theorem. Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{2}$$

where M and N have continuous first partial derivatives at all points (x,y) in a rectangular domain D. The differential equation (2) is exact if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

for all $(x,y) \in D$.

Example. Solve the following differential equations.
1)

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0.$$

Solution. We observe that the equation is exact since

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = 4x.$$

So, there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = 3x^2 + 4xy \qquad (3)$$

$$\frac{\partial F}{\partial y} = 2x^2 + 2y \qquad (4)$$

Thus, from (3), we get

$$F(x,y) = x^3 + 2x^2y + h(y) \qquad (5)$$

Using (4)and (5), we get

$$\frac{\partial F}{\partial y} = 2x^2 + 2y = 2x^2 + h'(y). \qquad (6)$$

From (6) we obtain $h'(y) = 2y$ and $h(y) = y^2 + c_1$. So, we obtain $F(x,y) = x^3 + 2x^2y + y^2 + c_1$ and the solution of given differential equation

$$x^3 + 2x^2y + y^2 = c.$$

2)

$$(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y) = 0,\ \ y(0) = 2.$$

3)

$$(ye^{xy}\tan x + e^{xy}\sec^2 x)dx + xe^{xy}\tan x dy = 0.$$

2.5. Integrating Factor

Consider the following differential equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

Definition. If the differential equation (1) is not exact but the differential equation

$$\lambda(x,y)M(x,y)dx + \lambda(x,y)N(x,y)dy = 0$$

is exact,then $\lambda(x,y)$ is called an integrating factor of the differential equation (1).

Theorem. *(i)*If

$$\frac{M_y - N_x}{N}$$

depends on x only, then

$$\lambda(x) = \exp \int \left(\frac{M_y - N_x}{N} \right) dx$$

is an integrating factor.
(ii)

$$\frac{M_y - N_x}{-M}$$

depends on y only, then

$$\lambda(y) = \exp \int \left(\frac{M_y - N_x}{-M} \right) dy$$

is an integrating factor.

Example. Solve the following differential equations.

1)

$$(x^2 + y^2 + x)dx + xydy = 0$$

Solution. Let us first observe that this equation is not exact. It is clear that $M_y = 2y$, $N_x = y$ and

$$\frac{M_y - N_x}{N} = \frac{1}{x}.$$

So there exists and integrating factor which depends on x. Integrating factor is calcuşated as

$$\lambda(x) = \exp \int \left(\frac{M_y - N_x}{N} \right) dx = x.$$

Multiplying equation by λ we obtain the equation

$$(x^3 + xy^2 + x^2)dx + x^2ydy = 0$$

which is exact. Now, we have to find the function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = x^3 + xy^2 + x^2 \qquad (i)$$

$$\frac{\partial F}{\partial y} = x^2y \qquad (ii)$$

Integrating (i), we get

$$F(x, y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + h(y) \qquad (iii)$$

Taking the derivative of (iii) with respect to y and using (ii), we obtain the solution of differential equation

$$3x^4 + 6x^2y^2 + 4x^3 = c$$

2)

$$(2xy^2 - 3y^3)dx + (7 - 3xy^2)dy = 0$$

3)

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + (3x^2y^2 + 4y)dy = 0$$