2.10. Lagrange-Clairaut Equations

Definition. A differential equation of the form

$$y = xf(y') + g(y') \tag{1}$$

is called Lagrange differential equation.

Let $p = y'$. Then from equation (1) we have

$$y = xf(p) + g(p) \tag{2}$$

Differentiating equation (2) with respect to x, we get

$$y' = p = f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx}$$

or

$$p - f(p) = (xf'(p) + g'(p))\,\frac{dp}{dx}.$$

Solving for p', we have

$$\frac{dp}{dx} = \frac{p - f(p)}{xf'(p) + g'(p)},\ p = p(x). \tag{3}$$

Let us assume that we can invert this function to find $x = x(p)$. Then from equatin (3) we get

$$\frac{dx}{dp} = \frac{xf'(p) + g'(p)}{p - f(p)},\ p - f(p) \neq 0$$

or

$$\frac{dx}{dp} - \frac{f'(p)}{p - f(p)}x = \frac{g'(p)}{p - f(p)} \tag{4}$$

which is a first order linear equation for $x(p)$.

Solving equation (4) it is obtained a family of solutions

$$x = F(p, c)$$

where c is an arbitrary integration constant. Using equation (2) one might be able to eliminate p to obtain a family of solutions of the Lagrange equation in the form

$$\varphi(x, y, c) = 0.$$

If it is not possible to eliminate p, then a parametric family of solutions with parameter p is obtained as

$$\begin{cases} x = F(p, c) \\ y = F(p, c) f(p) + g(p) \end{cases}$$

We assumed that $p - f(p) \neq 0$. However, there might also be solutions of Lagrange's equation for which $p - f(p) = 0$. Such solutions ara called singular solutions. To find singular solutions solve $p - f(p) = 0$ for p. Then substitute into equation (2).

Example. Solve the following differential equations.
1)

$$y = \frac{3}{2} x y' + e^{y'}$$

Solution. Denote $y' = p$. So, we have Lagrange equation

$$y = \frac{3}{2} x p + e^{p}.$$

Differentiating both sides of Lagrange equation with respect to x, we obtain

$$y' = p = \frac{3}{2} p + \frac{3}{2} x \frac{dp}{dx} + e^{p} \frac{dp}{dx}$$

or

$$-\frac{1}{2} p = \left(\frac{3}{2} x + e^{p} \right) \frac{dp}{dx}$$

Inverting this equality we obtain linear equation

$$\frac{dx}{dp} + \frac{3}{p} x = -\frac{2}{p} e^{p}, \ p \neq 0.$$

The integrating factor for linear equation is

$$\lambda(p) = p^{3}.$$

So, the general solution of linear equation is

$$x = -\frac{2e^{p}}{p^{3}} \left(p^{2} - 2p + 2 \right) + \frac{c}{p^{3}}.$$

Thus, the general solution of Lagrange equation in parametric form

$$\begin{cases} x = -\dfrac{2e^{p}}{p} + \dfrac{4e^{p}}{p^{2}} - \dfrac{4e^{p}}{p^{3}} + \dfrac{c}{p^{3}} \\ y = \dfrac{3}{2} x p + e^{p} \end{cases}$$

If $p = 0$, then singular solution is obtained as $y = 1$.

2)

$$y = x(1 + y') + y'^2$$

3)

$$y = x(p^2 + 2p) - (p^2 + 2p - 1)$$

Definition. The differential equation in the form

$$y = xy' + g(y') \tag{5}$$

is called Clairaut equation.

Letting $y' = p$ equation (5) is written as

$$y = xp + g(p).$$

The Clairaut equation is a particular case of the Lagrange equation. Differentiating both sides od Clairaut equation with respect x, we get

$$p = p + x\frac{dp}{dx} + g'(p)\frac{dp}{dx}$$

or

$$(x + g'(p))\frac{dp}{dx} = 0.$$

If $\dfrac{dp}{dx} = 0$, then we get $p = c$. So, we find the general solution

$$y = cx + g(c).$$

Solving $x + g'(p) =$ for p and substituting into Clairaut equation, the singular solution is obtained as

$$y = xp + g(p).$$

Example. Solve the following differential equations.
1)

$$y = xy' + (y')^2$$

Solution. Letting $y' = p$ we get Clairaut equation

$$y = xp + p^2$$

Differentiating both sides with respect to x, we have

$$(x + 2p)\frac{dp}{dx} = 0.$$

If $\dfrac{dp}{dx} = 0$, then $p = c$ and the general solution of Clairaut is

$$y = cx + c^2.$$

If $x + 2p = 0$, then $p = -\frac{x}{2}$ and singular solution is

$$y = -\frac{x^2}{4}.$$

2)

$$y = xy' + \sqrt{1 + (y')^2}$$

3)

$$y = xy' + \sin y'$$