CHAPTER 4. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

4.1. Basic Theory (Cont.)

We shall continue to investigate the properties of the following linear differential equations.

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x), \tag{1}$$

where a_0 is not identically zero.

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{2}$$

Theorem 3. Solutions $f_1, f_2, ..., f_n$ of equation (2) are linearly dependent on $[a, b]$ if and only if $W(f_1, f_2, ..., f_n) = 0$ for all $x \in [a, b]$.

Theorem 4. Let $f_1, f_2, ..., f_n$ be a set of n solutions of equation (2). Then either $W(f_1, f_2, ..., f_n) \equiv 0$ for all $x \in [a, b]$ or never zero in $[a, b]$.

Corollary 2. A necessary and sufficient condition that n solutions $f_1, f_2, ..., f_n$ of the nth order homogeneous linear differential equation (2) be linearly independent in $[a, b]$ is that

$$W(f_1, f_2, ..., f_n) \neq 0 \text{ for some } x \in [a, b].$$

Remark 1. This relationship between Wronskian and linear independence no longer holds if the functions are not solution of a homogeneous linear differential equation.

Corollary 1. Let f be a solution of the nth order homogeneous linear differential equation (2) such that

$$f(x_0) = f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0,\ x_0 \in [a, b].$$

Then $f(x) \equiv 0$ for all x on $[a, b]$.

Theorem 5. The nth order homogeneous linear differential equation (2) has n linearly independent solutions. Furthermore any other solutions of (2) can be written as a linear combination

$$c_1f_1 + c_2f_2 + ... + c_nf_n$$

of those n linearly independent solutions for suitable constants $c_1, c_2, ..., c_n$.

Definition 5. If $f_1, f_2, ..., f_m$ are n linearly independent solutions of the nth order homogeneous linear differential equation (2) on $[a, b]$, then the set $\{f_1, f_2, ..., f_n\}$ is called the fundamental set of the solutions of (2) and the function defined by

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x), \ x \in [a, b]$$

where $c_1, c_2, ..., c_n$ are arbitrary constants, is called the general solution of (2) on $[a, b]$.

Example 6. Let us consider the third order linear homogeneous differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6 = 0.$$

It is clear that the functions e^x, e^{2x}, e^{3x} are the solutions of the given differential equation. Moreover, these functions are linearly independent on every real interval since

$$W(e^x, e^{2x}, e^{3x})(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} \neq 0.$$

So, the fundamental set of solutions is $\{e^x, e^{2x}, e^{3x}\}$ and the general solution is

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

Theorem 6. Let g be any solution of the nonhomogeneous differential equation (1) and f be any solution of corresponding homogeneous differential equation (2). Then

$$f + g$$

is also a solution of equation (1).

Definition 6. The general solution of (2) is called the complementary function of equation (1). Any solution of equation (1) involving no arbitrary constants is called a particular solution of equation (1). If y_c is the complementary function, y_p is a particular solution then the solution

$$y_c + y_p$$

is called the general solution of (1).

Example 7. Let us consider the third order linear nonhomogeneous differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6 = e^{-x}.$$

By Example 6 we know that the complementary function is

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

Moreover it can be seen that a particular solution of given differential equation is

$$y_p = -\frac{1}{24}e^{-x}.$$

So, the general solution of given differential equation is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{1}{24}e^{-x}.$$

Example 8. Show that the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin 2x$$

is

$$y = c_1 + c_2e^{2x} - \frac{1}{5}e^x \sin 2x.$$