CHAPTER 4. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

4.2. Homogeneous Constant Coefficient Equations

In this section we consider homogeneous linear differential equations of the form

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}\frac{dy}{dx}+y=0, \tag{1}$$

where a_0 is not identically zero and $a_0, a_1, ...a_n$ are real constants.

Assume that $y=e^{mx}$ is a solution of equation (1). Then we have

$$e^{mx}\left(a_0m^n+a_1m^{n-1}+...+a_{n-1}m+a_n\right)\equiv 0.$$

Since $e^{mx}\neq 0$, we obtain the equation in the unknown m

$$a_0m^n+a_1m^{n-1}+...+a_{n-1}m+a_n=0 \tag{2}$$

which is called the characteristic equation of the given equation (1). Now, we have three cases.

Theorem 1. If the characteristic equation (2) has n distinct real roots $m_1, m_2, ..., m_n$, then the general solution of (1) is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+...+c_ne^{m_nx},$$

where $c_1, c_2, ..., c_n$ are arbitrary constants.

Example 1. Find the general solution of the following differential equation

$$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-2\frac{dy}{dx}=0.$$

Solution. The characteristic equation of given differential equation is

$$m^3-m^2-2m=0.$$

Hence, we obtain that $m_1=0,\ m_2=2$ and $m_3=-1$. The roots are real and distinct. So, the general solution is

$$y=c_1+c_2e^{2x}+c_3e^{-x}.$$

Theorem 2. **(i)** If the characteristic equation (2) has the real root m occuring k times and the remainig roots are distinct and real, then the general solution of the equation (1) is

$$y=\left(c_1+c_2x+c_3x^2+...+c_kx^{k-1}\right)e^{mx}+c_{k+1}e^{m_{m+1}x}+...+c_ne^{m_nx}.$$

Example 2. Find the general solution of the following differential equation

$$\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0.$$

Solution. The characteristic equation of given differential equation is

$$m^4 - 5m^3 + 6m^2 + 4m - 8 = 0$$

Hence, we obtain that $m_1 = m_2 = m_3 = 2$ and $m_4 = -1$. So, the general solution is

$$y = e^{2x}(c_1 + c_2x + c_3x^2) + c_4e^{-x}.$$

Theorem 3. (i) If the characteristic equation (2) has the conjugate complex roots $a \pm ib$, then the corresponding part of the general solution is

$$y = e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right).$$

(ii) If $a \pm ib$ are each $k-$fold roots of the characteristic equation (2), then the corresponding part of the general solution is

$$y = e^{ax}\left[\left(c_1 + c_2x + ... + c_kx^{k-1}\right)\cos bx + \left(c_{k+1} + c_{k+2}x + ... + c_{2k}x^{k-1}\right)\sin bx\right].$$

Example 3. Solve the initial value problem

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0,\ y(0) = -3,\ y'(0) = -1.$$

Solution. The characteristic equation of given differential equation is

$$m^2 - 6m + 25 = 0.$$

So, the roots are $m_{1,2} = 3 \pm 4i$ and the general solution of given differential equation is

$$e^{3x}\left(c_1 \cos 4x + c_2 \sin 4x\right).$$

Applying the initial condition $y(0) = -3$, we get $c_1 = -3$. From the other condition $y'(0) = -1$, we obtain $c_2 = 2$. So, the solution of given initial value problem is

$$e^{3x}\left(-3\cos 4x + 2\sin 4x\right).$$

Example. Find the general solutions of following differential equations.

1)

$$\frac{d^4y}{dx^4} - 9\frac{d^2y}{dx^2} + 20y = 0$$

2)

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

3)

$$\frac{d^6y}{dx^6} - 5\frac{d^4y}{dx^4} + 16\frac{d^3y}{dx^3} + 36\frac{d^2y}{dx^2} - 16\frac{dy}{dx} - 32y = 0$$