CHAPTER 4. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

4.3. The Method of Undetermined Coefficients

In this section we consider nonhomogeneous linear differential equations of the form

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}\frac{dy}{dx}+y=f(x), \tag{1}$$

where a_0 is not identically zero and $a_0, a_1, ...a_n$ are real constants.

Recall that the general solution of equation (1) is

$$y=y_c+y_p,$$

where y_c is the complementary function, that is, the general solution pf corresponding homogeneous equation, y_p is a particular solution of equation (1).

Now, we consider methods of determining a particular solutions.

Definition 1. If a function defined by one of the following forms or defined as a finite product of two or more functions of these types, then the function is called a UC function:

(i) x^n, where n is a positive integer or zero.

(ii) e^{ax}, where a is a constant.

(iii) $\sin(ax+b)$ ar $\cos(ax+b)$, where a and b are constants, $a \neq 0$.

Remark 1. The method of undetermined coefficients applied when the nonhomogeneous term $f(x)$ in the differential equation (1) is a finite linear combination of UC functions.

Definition 2. Consider a UC function f. The set of functions consisting of f itself and successive derivatives of f is called UC set of f.

Example 1. $f(x)=x^3$ is a UC function. UC set:

$$S=\{x^3, x^2, x, 1\}$$

Method:

Step 1. Solve the homogeneous equation and write fundamental set of solutions.

Step 2. Find UC set of f.

Step 3. If the UC set of f includes one or more members of fundamental set of solutions, then multiply each member of S by the lowest positive integer power of x. So, the new set does not inclede any member of fundamental set of solutions.

Step 4. The linear combination of the members of S_1 is the form of particular solution.

Example 2. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = e^x - 10\sin x$$

Solution. The characteristic equation of the corresponding differential equation is

$$m^2 - 2m - 3 = 0.$$

So, the roots are $m_1 = 3,\ m_2 = -1$ and the complementary function is

$$y_c = c_1e^{3x} + c_2e^{-x}.$$

Then the fundamental set of solutions is

$$FSS = \{e^{3x}, e^{-x}\}.$$

The UC set of e^x is $S_1 = \{e^x\}$; the UC set of $\sin x$ is $S_2 = \{\sin x, \cos x\}$. So, the UC set is

$$S = S_1 \cup S_2 = \{e^x, \sin x, \cos x\}.$$

Since the UC set S does not include any member of fundamental set, the particular solution may be in the form

$$y_p = Ae^x + B\sin x + C\cos x,$$

where the constants A, B and C will be determined. Now, substituting y_p and its derivatives into given differential equation we have

$$-4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x \equiv 2e^x - 10\sin x.$$

Equating coefficients of these like terms, we obtain the equations

$$-4A = 2,\ \ -4B + 2C = -10,\ \ -4C - 2B = 0.$$

So, we have

$$A = -\frac{1}{2},\ B = 2,\ C = -1.$$

Hence the particular solution is

$$y_p = -\frac{1}{2}e^x + 2\sin x - \cos x$$

and the general solution is

$$y = c_1e^{3x} + c_2e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x.$$

Example. Find the general solutions of following differential equations.

1)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x + 1$$

2)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = xe^x$$

3)

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 4y = 8x + 8 + 6e^{-x}.$$