CHAPTER 4. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

4.5. The Method of Variation of Parameters

In this section we consider a general method of determining a particular solution of the equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x), \tag{1}$$

where the functions $a_0(x), a_1(x)$ and $a_2(x)$ are continuous on the interval $[a, b]$.

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0. \tag{2}$$

Then the complementary function of equation (2) is

$$y_c = c_1y_1(x) + c_2y_2(x)$$

The procedure in the method of variation of parameters is to replace the arbitrary constants c_1 and c_2 by the functions $v_1(x)$ and $v_2(x)$ which will be determined so that the resulting function

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{3}$$

will be a particular solution of equation (1). It can be seen that to determine $v_1(x)$ and $v_2(x)$, we have the following system of equations

$$\begin{aligned} v_1'(x)y_1(x) + v_2'(x)y_2(x) &= 0 \\ v_1'(x)y_1'(x) + v_2'(x)y_2'(x) &= \frac{f(x)}{a_0(x)} \end{aligned} \tag{4}$$

for unknown functions v_1' and v_2'.

Since y_1 and y_2 are linearly independent solutions of equation (2), the determinant of coefficients of this system

$$\begin{aligned} W\left(y_1(x), y_1(x)\right) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= y_1y_2' - y_2y_1' \\ &\neq 0. \end{aligned}$$

By Cramer's Rule, the solution of system (4) is obtained as

$$v_1'(x) = \frac{1}{W\left(y_1(x), y_1(x)\right)} \begin{vmatrix} 0 & y_2(x) \\ \frac{f(x)}{a_0(x)} & y_2'(x) \end{vmatrix}$$

$$v_2'(x) = \frac{1}{W(y_1(x), y_1(x))} \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & \frac{f(x)}{a_0(x)} \end{vmatrix}$$

Integrating $v_1'(x)$ and $v_2'(x)$, we can find $v_1(x)$ and $v_2(x)$. So, the particular solution of equation (1) is

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

Remark 1. This method can be apply higher order differential equations. Let us consider

$$\frac{d^3y}{dx^3} + a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x)$$

The particular solution of this equation will be in the form

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x) + v_3(x)y_3(x),$$

where y_1, y_2 and y_3 are linearly independent solutions of corresponding homogeneous equation. Similarly, from the following system v_1', v_2' and v_3' can be found:

$$\begin{aligned} v_1'y_1 + v_2'y_2 + v_3'y_3 &= 0 \\ v_1'y_1' + v_2'y_2' + v_3'y_3' &= 0 \\ v_1'y_1'' + v_2'y_2'' + v_3'y_3'' &= f(x) \end{aligned}$$

Example 1. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \cos ecx$$

Solution. It can be easily seen that the complementary function is

$$y_c = c_1 \cos x + c_2 \sin x$$

So, the particular solution is in the form

$$y_p = v_1(x)\cos x + v_2(x)\sin x.$$

v_1 and v_2 should be satisfy the following system:

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0 \\ v_1'(-\sin x) + v_2'(\cos x) &= \frac{1}{\sin x} \end{aligned}$$

The determinanf of the coefficients of this system is

$$\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0.$$

So,

$$v_1'(x) = \begin{vmatrix} 0 & \sin x \\ \frac{1}{\sin x} & \cos x \end{vmatrix} = -1$$

and

$$v_2'(x) = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{1}{\sin x} \end{vmatrix} = \frac{\cos x}{\sin x}$$

Integrating these equations we obtain

$$v_1(x) = -x$$

and

$$v_2(x) = \ln(\sin x).$$

Thus the particular solution is found as

$$y_p = -x \cos x + \ln(\sin x) \sin x$$

The general solution of given differential equation is

$$y = c_1 \cos(x) + c_2 \sin x - x \cos x + \ln(\sin x) \sin x.$$

Example. Find the general solutions of following differential equations.

1)

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \frac{1}{1 + e^{-x}}$$

2)

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \frac{1}{\cos x}$$