PHY401
Electromagnetic Theory I

Electromagnetic Waves in Conductors

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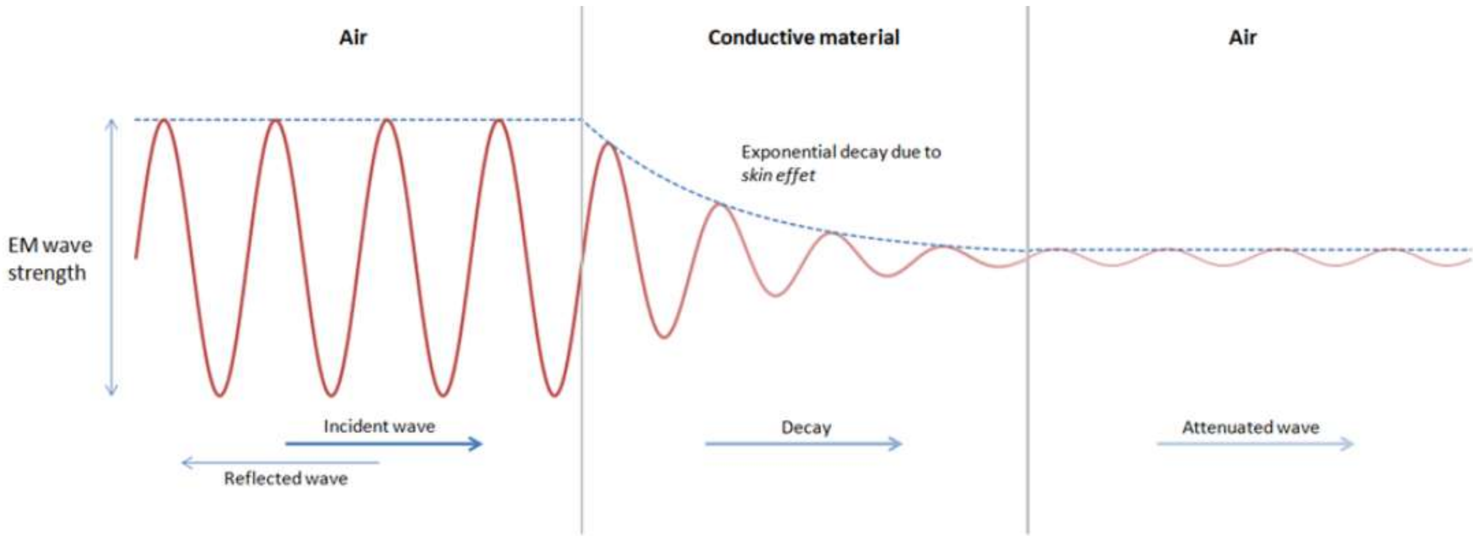
Chapter 9. Electromagnetic Waves and Its Applications



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9.4 Electromagnetic Waves and Its Applications
9.4.1 Electromagnetic Waves in Conductors

When wave propagates through vacuum or insulating materials such as glass or teflon, assuming no free charge and no free current is reasonable. But in conductive media such as metal or plasma, the free charge and free current are generally not zero.

$$free\ charge\ density\ \rho_f \neq 0,\ free\ current\ density\ J_f \neq 0$$

$$\vec{J}_f = \sigma \vec{E}$$

With this, Maxwell's equations for linear media assume the form:

$$(i)\ \vec{\nabla}.\vec{E} = \frac{\rho_f}{\varepsilon} \qquad (iii)\ \vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

$$(ii)\ \vec{\nabla}.\vec{B} = 0 \qquad (iv)\ \vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial E}{\partial t} + \mu J_f$$

The continuity equation for free charge: $$\frac{\partial \rho_f}{\partial t} = -\nabla . J_f$$

Remember that derivation of the continuity equation for free charge was:

The divergence of the curl of any vector field is always zero:

$$\nabla \cdot \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) = \nabla \cdot (\nabla \times \mathbf{H}) = 0$$

$$\frac{\partial (\nabla \cdot \mathbf{D})}{\partial t} = -\nabla \cdot \mathbf{J}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_V}{\partial t}$$

1. $\nabla \cdot \mathbf{D} = \rho_V$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
4. $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

$$\frac{\partial \rho_f}{\partial t} = -\nabla . J_f$$

$$\frac{\partial \rho_f}{\partial t} = -\sigma(\nabla \cdot \mathbf{E}) = -\sigma \frac{\rho_f}{\varepsilon} = -\frac{\sigma}{\varepsilon}\rho_f$$

For a homogeneous linear medium it follows that

$$\rho_f(t) = e^{-(\sigma/\varepsilon)t}\rho_f(0)$$

Thus any initial free charge $\rho_f(t)$ dissipates in a characteristic time $\tau = \varepsilon/\sigma$

This reflects the familiar fact that if you put some free charge on a conductor, it will flow out to the edges.

- Perfect conductor $\sigma = \infty$, $\tau = 0$ (10^{-14} s for copper so it is a good conductor)
- Poor conductor $\tau >> \varepsilon/\sigma$ (vacuum, glass, pure water, etc.)

Electromagnetic Waves in Conductors-Omitting Transient Effect

We'll wait for any accumulated free charge to disappear. From then on, $\rho_f = 0$, and we have

i) $\nabla \cdot \mathbf{E} = 0$ iii) $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

ii) $\nabla \cdot \mathbf{B} = 0$

iv) $\nabla \times \mathbf{B} = \mu\varepsilon \dfrac{\partial \mathbf{E}}{\partial t} + \mu\sigma\mathbf{E}$

Applying the curl to (iii) and using (i),

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

$=0$

Then using (iv)

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$

Wave equations for ***E*** and ***B***,

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla^2 \mathbf{B} = \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}$$

These equations still admit plane-wave solutionsv

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)} \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)}$$

Note this time the wave number **k** is complex.

$$\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

$$\tilde{k} = k + i\kappa$$

$$k \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right] \quad \kappa \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]$$

The imaginary part of $\tilde{k}$ results in attenuation of the wave (decreasing amplitude with increasing z):

$$\tilde{E}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

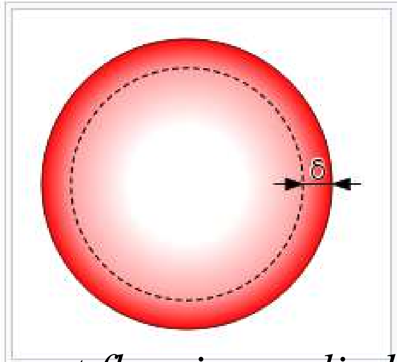
$$\tilde{B}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

Skin depth, *d*: $d = 1/\kappa$

$d = 2/(\sigma)\sqrt{\varepsilon/\mu}$ → **poor conductor**

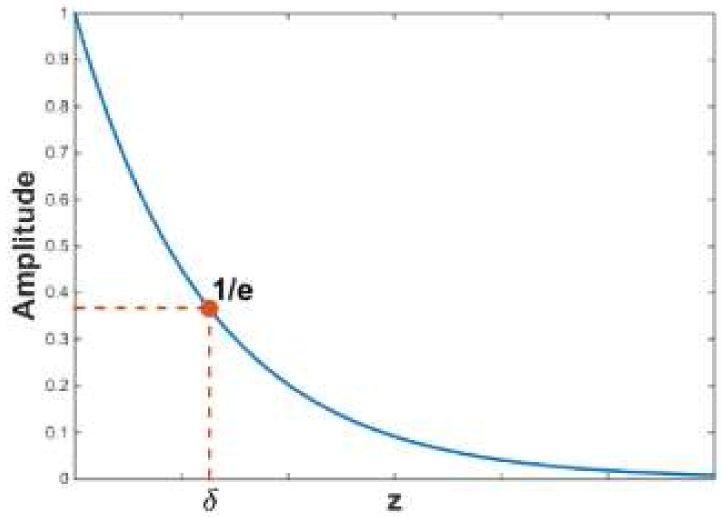
$d = \lambda/(2\pi)$ → **good conductor**

The electric current flows mainly between the outer edge and skin depth of the conductor.



Current flow in a cylindrical conductor

The distance it takes to reduce the amplitude by a factor of **1/e** (about a third) is called the **skin depth.** It is a measure of how far the wave penetrates into the conductor. Skin depth is so large in poor conductors, so they are transparent in visible light (such as glass, water, etc.). Meanwhile, the real part of k determines the wavelength, the propagation speed, and the index of refraction, in the usual way.



Skin depth for various materials

Type	σ	μ_r	ϵ_r	δ (1Hz)	δ (1kHz)
Air	0 S/m	1	1	∞	∞
Sea Water	3.3 S/m	1	80	277 m	8.76 m
Igneous	10^{-4} S/m	1	5	50,300 m	1,590 m
Sedimentary (dry)	10^{-3} S/m	1	4	15,900 m	500 m
Sedimentary (wet)	10^{-2} S/m	1	25	5,000 m	160 m
Sulphide Skarn	10^{2} S/m	1	5	50 m	1.6 m
Magnetite Skarn	10^{2} S/m	2	5	36 m	1.1 m

Phase Difference Between Electric and Magnetic Fields

Suppose that **E** is polarized along the *x* direction:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega} \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

Like any complex number, *k* can be expressed in terms of its modulus and phase

$$\tilde{k} = k + i\kappa = Ke^{i\phi}$$

where

$$K \equiv \sqrt{k^2+\kappa^2} = \omega\sqrt{\varepsilon\mu\sqrt{1+(\frac{\sigma}{\varepsilon\omega})^2}} \qquad \phi \equiv \tan^{-1}(\kappa/k)$$

$\widetilde{E}$ *and* $\widetilde{B}$ *are related by* $$B_0 e^{i\delta_B} = \frac{Ke^{i\phi}}{\omega} E_0 e^{i\delta_E}$$

The magnetic field *lags behind* the electric field.

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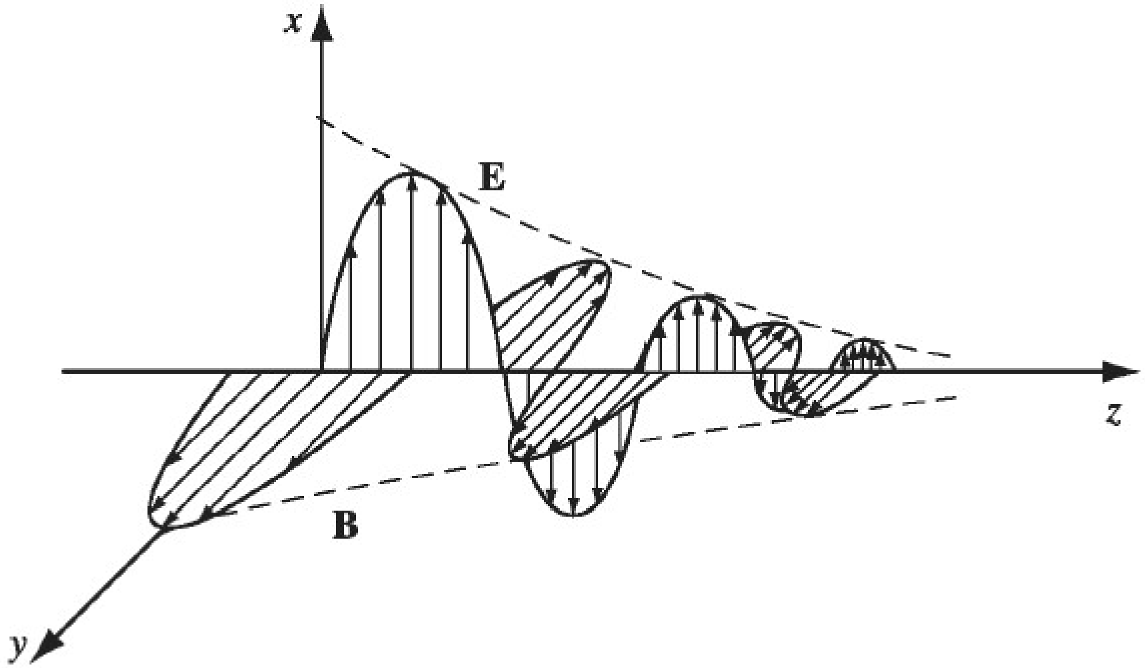


The real electric and magnetic fields are:

$$\mathbf{E}(z,t) = E_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E)\hat{\mathbf{x}}$$

$$\mathbf{B}(z,t) = \frac{K}{\omega} E_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi)\hat{\mathbf{y}}$$

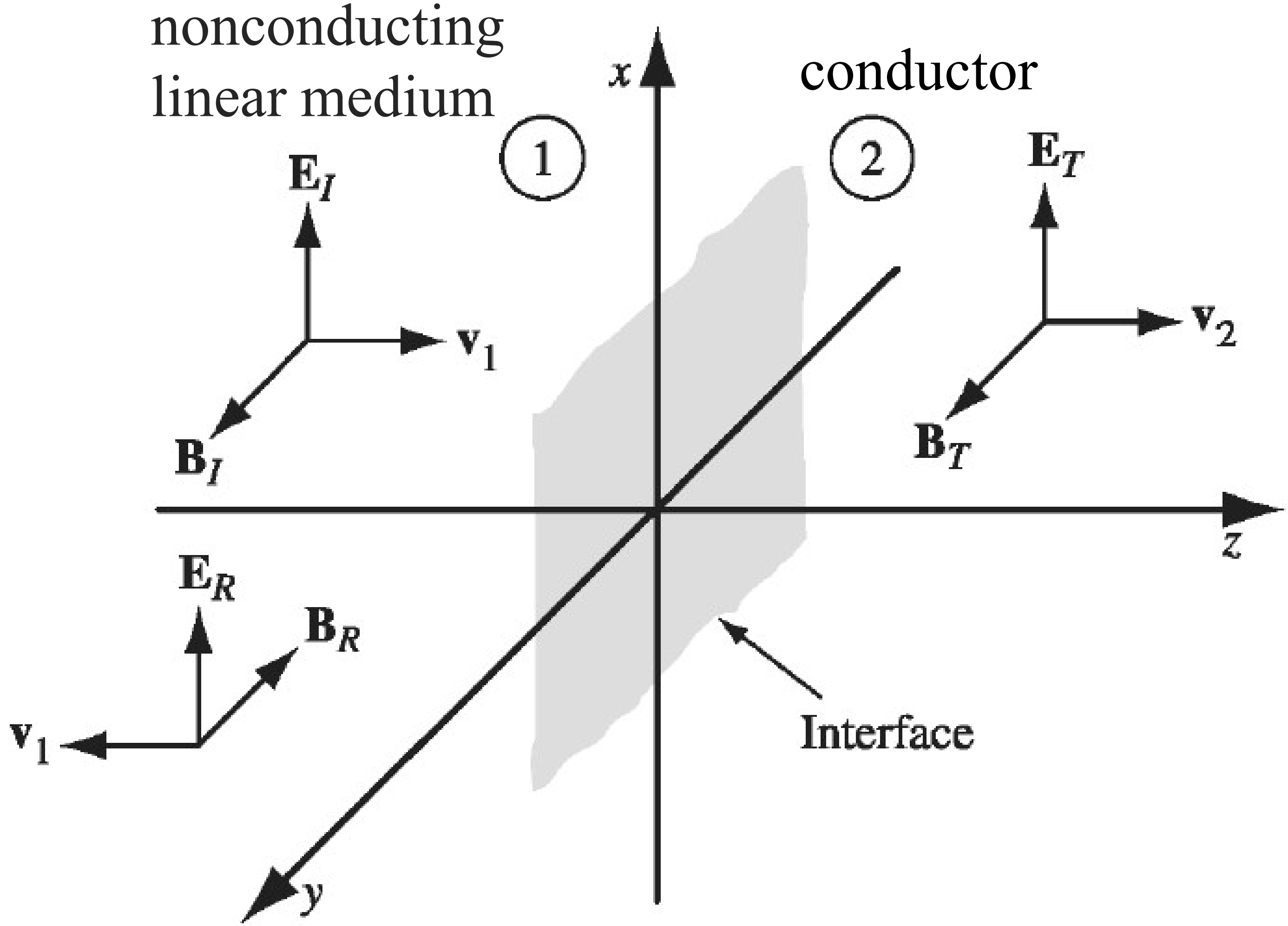
9.4.2 Reflection at a Conducting Surface

The boundary conditions we used to analyze reflection and refraction at an interface between two dielectrics do not hold in the presence of free charges and currents. Instead, we have the more general relations (Eq. 7.64)

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

where σ_f (not to be confused with conductivity) is the free surface charge, $\mathbf{K_f}$ is the free surface current, and **n** (not to be confused with the polarization of the wave) is a unit vector perpendicular to the surface, pointing from medium (2) into medium (1).


nonconducting
linear medium
conductor
x
1
2
$\mathbf{E}_I$
$\mathbf{v}_1$
$\mathbf{B}_I$
$\mathbf{E}_T$
$\mathbf{v}_2$
$\mathbf{B}_T$
z
$\mathbf{E}_R$
$\mathbf{B}_R$
$\mathbf{v}_1$
Interface
y

The incident wave:

$$\tilde{E}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x} \qquad \tilde{B}_I(z,t) = \frac{1}{\vartheta_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflected wave:

$$\tilde{E}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x} \qquad \tilde{B}_R(z,t) = -\frac{1}{\vartheta_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{y}$$

Transmitted wave:

$$\tilde{E}_T(z,t) = \tilde{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)} \hat{x} \qquad \tilde{B}_T(z,t) = \frac{\tilde{k}_2}{\omega} e^{-\kappa z} e^{i(k_2 z - \omega t)} \hat{y}$$

Transmission is attenuated, k_2 is a complex quantity.

$B^{\perp} = 0$

$E^{\perp} = 0$ which yields that $\sigma_f = 0$

$E_1^{\mathrm{II}} - E_2^{\mathrm{II}} = 0$ due to 1st boundary condition, so free surface current $K_f=0$