# PHY401
# Electromagnetic Theory I

## Scalar and Vector Potentials and Gauge Transformations, Coulomb and Lorenz Gauge

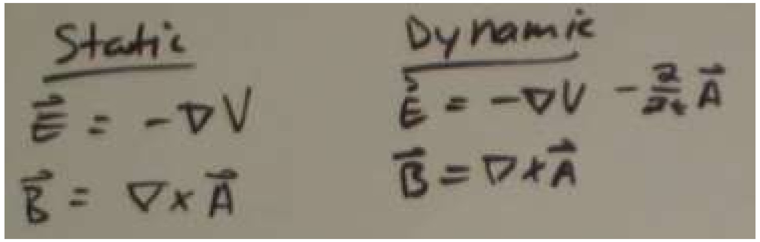
Assoc. Prof. Dr. Fulya Bagci

Department of Physics Engineering/Ankara University

fbagci@eng.ankara.edu.tr

# Contents




Static
$\vec{E} = -\nabla V$
$\vec{B} = \nabla \times \vec{A}$
Dynamic
$\vec{E} = -\nabla V - \frac{\partial}{\partial t}\vec{A}$
$\vec{B} = \nabla \times \vec{A}$

# 10.1.1 Scalar and Vector Potentials

*Given ρ(r,t) and J(r,t), what are the fields E(r,t) and B(r,t)?*

In the static case -> Coulomb's law and the Biot-Savart law

In the dynamic case? -> ?

The fields have to be represented in terms of potentials.

$$i)\ \nabla.\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \quad iii)\ \nabla\times\boldsymbol{E} = 0$$

$$ii)\ \nabla.\boldsymbol{B} = 0 \quad iv)\ \nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}$$

In electrostatics, $\vec{\nabla}\times\vec{E} = 0 \rightarrow \vec{E} = -\nabla V \quad \vec{\nabla}\vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$

In electrodynamics $\vec{\nabla}\times\vec{E} \neq 0 \quad \vec{\nabla}\vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right)$$

$$\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\nabla V \rightarrow \boxed{\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}}$$

This function **A** is given the name "**vector potential**"

Doç.Dr. Fulya Bağcı

**Example 10.1**. Find the charge and current distributions that would give rise to the potentials

$$V = 0,\ \mathbf{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2\,\hat{\mathbf{z}} & \text{for } |x| < ct \\ 0 & \text{for } |x| > ct \end{cases}$$

Solution:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\hat{z}$$

$$\vec{\nabla}\vec{E} = 0 \rightarrow \rho = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu_0 k}{4c}(ct - |x|)^2 \end{vmatrix}$$

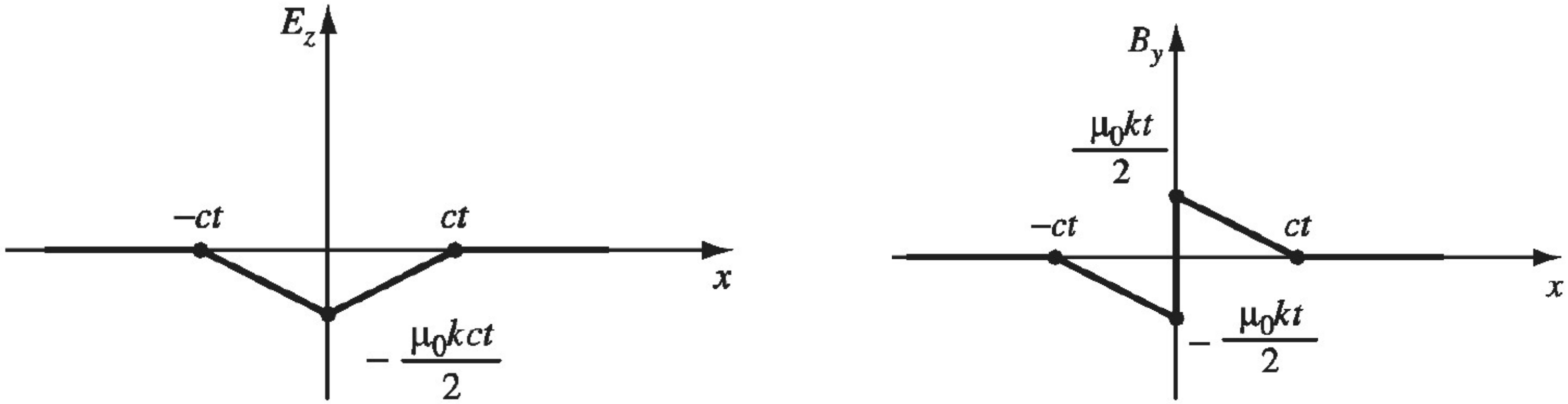
$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{\mu_0 k}{2}(ct - |x|) \end{vmatrix} = \mp \frac{\mu_0 k}{2}\hat{y}$$

$$\vec{B} = \pm \frac{\mu_0 k}{2c}(ct - |x|)\hat{y}$$

$$\vec{\nabla}\vec{B} = 0$$

Notice that **B** has a discontinuity at $x = 0$.

$$\frac{\partial B}{\partial t} = \pm \frac{\mu_0 k}{2}\hat{y} \qquad \frac{\partial E}{\partial t} = -\frac{\mu_0 k}{2}c\hat{z}$$




$$\mathbf{J} = -\frac{1}{\mu_0}\left(\nabla^2\mathbf{A} - \mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\right) + \frac{1}{\mu_0}\nabla(\nabla\cdot\mathbf{A})$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

$$\nabla^2\mathbf{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)A_z\hat{\mathbf{z}} = \frac{\mu_0 k}{4c}\hat{\mathbf{z}} \qquad \mathbf{J} = 0$$

$$-\mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\varepsilon_0\frac{\mu_0 k}{4c}c^2\hat{\mathbf{z}} = \frac{\mu_0 k}{4c}\hat{\mathbf{z}}$$

Since the volume charge density and current density are both zero, where are the electric and magnetic fields from?

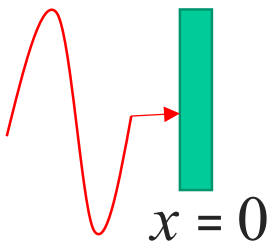
Remember the boundary condition (iv) in Eq. 7.64

$$\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

$$kt\,\hat{y} = \vec{K} \times \hat{x}$$

$$\vec{K} = kt\,\hat{z}$$

We have here a uniform surface current flowing in the z direction over the plane $x$ = 0. Notice that the news travels out (in both directions) at the speed of light: for points |x| > $ct$ the message ("current is now flowing") has not yet arrived, so the fields are zero.



## 10.2 Gauge Transformations

We are free to impose extra conditions on $V$ and $\boldsymbol{A}$, when nothing happens to **E and B.** Let's work out precisely what this **gauge freedom** entails.

$$A' = A + \alpha \qquad V = V' + \beta$$

Since two A's give the same **B,** their curls must be equal, and hence,

$$\nabla \times \alpha = 0 \rightarrow \alpha = \nabla\lambda$$

The two potentials also give the same **E,** so

$$\nabla \cdot \beta + \frac{\partial \alpha}{\partial t} = 0 \rightarrow \nabla \cdot \left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0 \rightarrow \beta = -\frac{\partial \lambda}{\partial t} + k(t)$$

We might as well absorb $k(t)$ into $\boldsymbol{A}$.

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\lambda \qquad V' = V - \frac{\partial \lambda}{\partial t}$$

Such changes in $V$ and $\boldsymbol{A}$ are called gauge transformations.
They can be exploited to adjust the divergence of $\boldsymbol{A}$.

# 10.1.3 Coulomb Gauge and Lorenz Gauge

**The Coulomb Gauge:** In the Coulomb gauge, we pick $\nabla \cdot \mathbf{A} = 0$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho \xrightarrow{\nabla \cdot \mathbf{A} = 0} \nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

This is Poisson's equation, and we already know how to solve it: setting $V = 0$ at infinity.

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}', t)}{r} d\tau' \quad \text{(setting } V=0 \text{ at infinity)}$$

$$\left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \xrightarrow{\nabla \cdot \mathbf{A} = 0} \nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right)$$

**The Lorentz Gauge:** In the Lorenz gauge, we pick $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$

With this, $\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$

Meanwhile, the differential equation for $V$ becomes $\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho$

Lorenz gauge treats $V$ and A with the same differential operator called the ***d'Alembertian***. In the Lorenz gauge, $V$ and A satisfy the **inhomogeneous wave equation**, with a "source" term on the right.

$$\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \equiv \Box^2$$

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

This choice of Coulomb gauge is mostly appropriate for the study of radiation problems and allow us to write down the Poisson’s equation for the scalar potential.

For a covariant treatment of the electrodynamics the Lorentz gauge choice is preferred. Then the uncoupled differential equations turn out to be inhomogenous wave equations. Using d’Alembertian operator (denoted by $\Box$), the differential equations for the potentials become simpler.

*When you get to dynamics, you'll see that potentials are much easier to work with than fields.*