PHY401
Electromagnetic Theory I

Waves in One Dimension

Assoc. Prof. Dr. Fulya Bagci

Department of Physics
Engineering/Ankara University

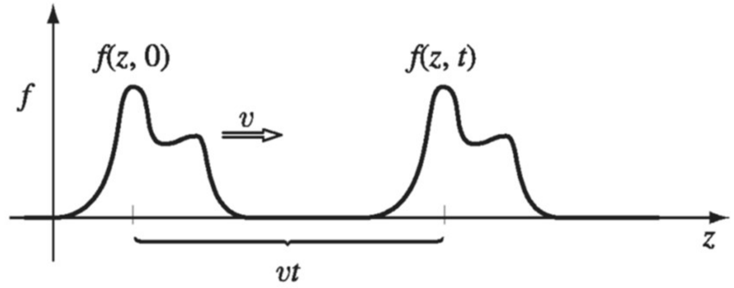
fbagci@eng.ankara.edu.tr

Contents

9.1 Waves in One Dimension

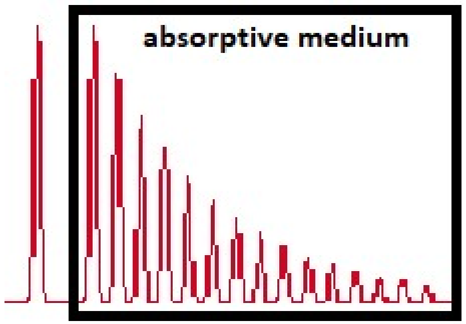
9.1.1 The Wave Equation

A wave is disturbance of a continuous medium that propagates with a fixed shape at constant velocity.



Lets start from the simplest case to write the **wave equation**. Obviously there can be absorption, dispersion effects and the shape of the wave is affected.

In the presence of absorption, the wave will diminish in size as it moves.



$$A = A_0 e^{-\alpha z}$$

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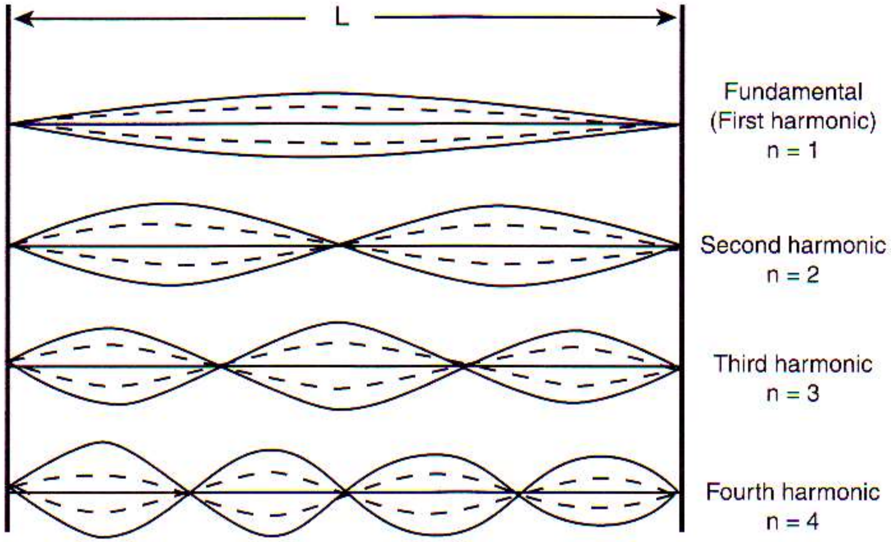
If the medium is dispersive different frequencies travel at different speeds.

Red light travel faster than blue light

Standing waves (combination of two waves moving in opposite directions, each having the same amplitude and frequency) do not propagate.



Light wave can propagate in vacuum.

How to represent a "wave" mathematically?

Once at t=0, and again at some later time t each point on the wave form simply shifts to the right by an amount vt, where v is the velocity

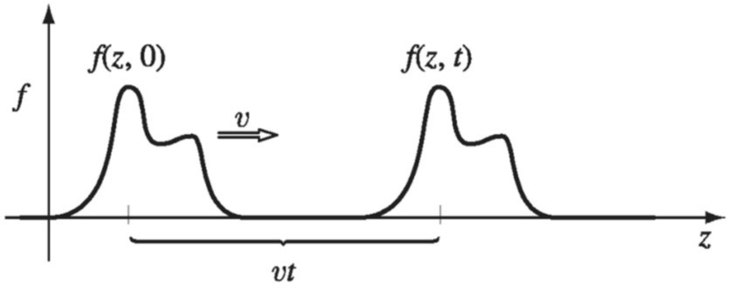
Initial shape: $f(z,0) = g(z)$

Subsequent form: $f(z,t) = ?$

A displacement at point z at time t is the same as the displacement a distance Vt to the left back at time t=0:

$$f(z,t) = f(z - vt, 0)$$

$$= g(z\text{-}vt)$$



The function $f(z,t)$ depends on them only in the very special combination $z\text{-}vt$ (or $z+vt$)

Examples

$$f_1(z,t) = Ae^{-b(z-vt)^2}$$

$$f_2(z,t) = A\sin[b(z-vt)]$$

$$f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$

Not a wave:

$$f_4(z,t) = Ae^{-b(z^2+vt)}$$

Not a wave:

$$f(z,t) = A\sin(bz)\cos(bvt)^3$$

A standing wave:

$$f_5(z,t) = A\sin(bz)\cos(bvt)$$
$$= \frac{A}{2}[\sin(b(z+vt)) + \sin(b(z-vt))]$$

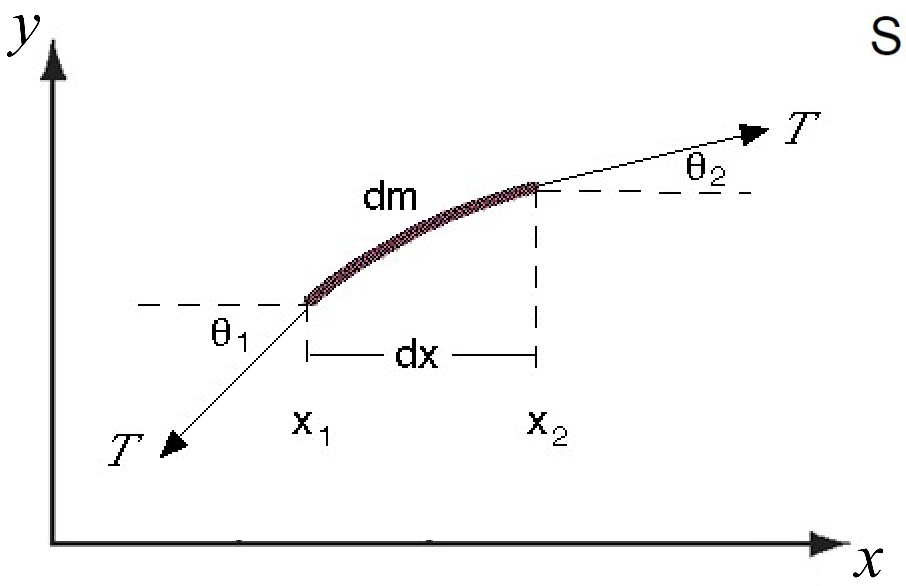
The Wave Equation of a String

Differential Form:	**Force = change of momentum with change of time**	$F = \frac{d(mv)}{dt}$
With mass constant:	**Force = mass X acceleration**	$F = m a$

Let's apply Newton's second law in the vertical *y*-direction.

μ: mass per unit length

$$F_y = T\sin(\theta_2) - T\sin\theta_1 = (\mu dx)\frac{\partial^2 y}{\partial t^2}$$



Small angle approximation: $\sin\theta \approx \tan\theta = \frac{\partial y}{\partial x}$

θ_1 and $\theta_2 < 15°$

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x=x_2} - \left.\frac{\partial y}{\partial x}\right|_{x=x_1}\right) = (\mu dx)\frac{\partial^2 y}{\partial t^2}$$

$$\frac{T\left(\left.\frac{\partial y}{\partial x}\right|_{x=x_2} - \left.\frac{\partial y}{\partial x}\right|_{x=x_1}\right)}{dx} = \mu\frac{\partial^2 y}{\partial t^2}$$

$$\frac{T\left(\left.\frac{\partial y}{\partial x}\right|_{x=x_2} - \left.\frac{\partial y}{\partial x}\right|_{x=x_1}\right)}{dx} = \mu \frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad T\frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

Acceleration is proportional to tension *T*, inversely proportional to μ.

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}$$

Speed of a wave in a stretched string:

$$\vartheta = \sqrt{\frac{T}{\mu}}$$

Classical wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\vartheta^2}\frac{\partial^2 y}{\partial t^2}$$

Note that y is a function dependent on x.

The most general solution to the wave equation is the sum of a wave to the right and a wave to the left:

$$f(z,t) = g(z - vt) + h(z + vt)$$

$\longrightarrow z \qquad -z \longleftarrow$

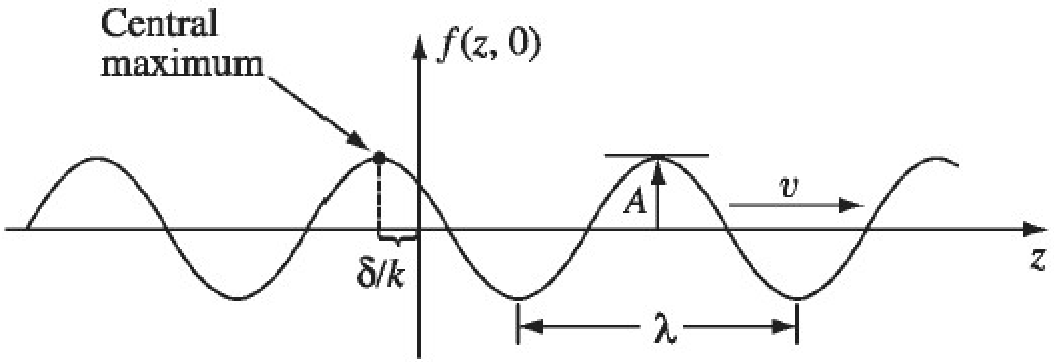
Sinusoidal Waves

wave speed

$$f(z,t) = A\cos[k(z - vt) + \delta]$$

amplitude   wave number   phase constant

Amplitude: Maximum displacement from equilibrum
Phase: The argument of the cosine



$$f(z,t) = A\cos[k(z - vt) + \delta] = A\cos(kz - \omega t + \delta)$$

$$k = \frac{2\pi}{\lambda}, \quad \lambda\text{: wave length}$$

$$\omega = kv = 2\pi\frac{v}{\lambda} = 2\pi f$$

ω: angular frequency

f: frequency

Angular frequency (in radians) is larger than regular **frequency** (in Hz) by a factor of 2π

(ii) Complex notation

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

$$\mathrm{Re}[Ae^{i(kz-\omega t+\delta)}] = A\cos[k(z-vt)+\delta]$$

Wave equation (sinusoidal)

$$\mathrm{Re}[A^{i\delta}e^{i(kz-\omega t)}] = \mathrm{Re}[\tilde{A}e^{i(kz-\omega t)}]$$

$$\tilde{A} = Ae^{i\delta}$$

complex wave function

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)]$$

The advantage of the complex notation is that exponentials are *much easier to manipulate* than sines and cosines.

Example 9.1

Suppose you combine two sinusoidal waves, you simply add the corresponding complex wave functions and then take the real part.

$$f_3 = f_1 + f_2 = \mathrm{Re}[\tilde{f}_1] + \mathrm{Re}[\tilde{f}_2] = \mathrm{Re}[\tilde{f}_1 + \tilde{f}_2] = \mathrm{Re}[\tilde{f}_3]$$

In particular, if they have the same frequency and wave number, it is very easy.

$$\tilde{f}_3 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = \tilde{A}_3 e^{i(kz-\omega t)}$$

where $A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}$

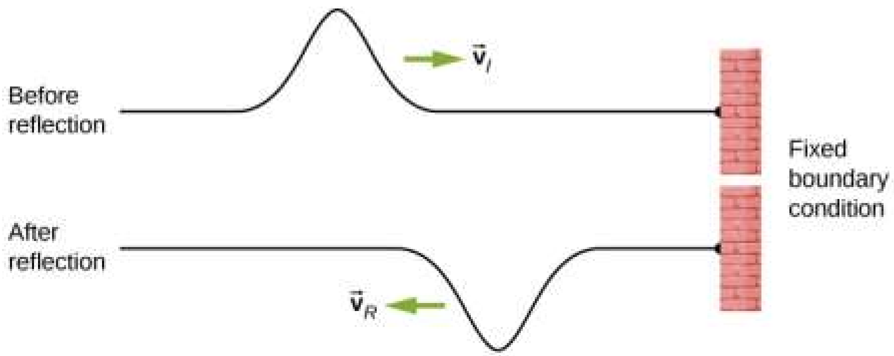
Try Prob. 9.3.

Try solving Prob. 9.3 without complex notation to see the difference.

Boundary Conditions: Reflection and Transmission

When a wave propagates through a medium, it reflects when it encounters the boundary of the medium. Waves will react differently if the boundary of the medium is fixed in place or free to move.

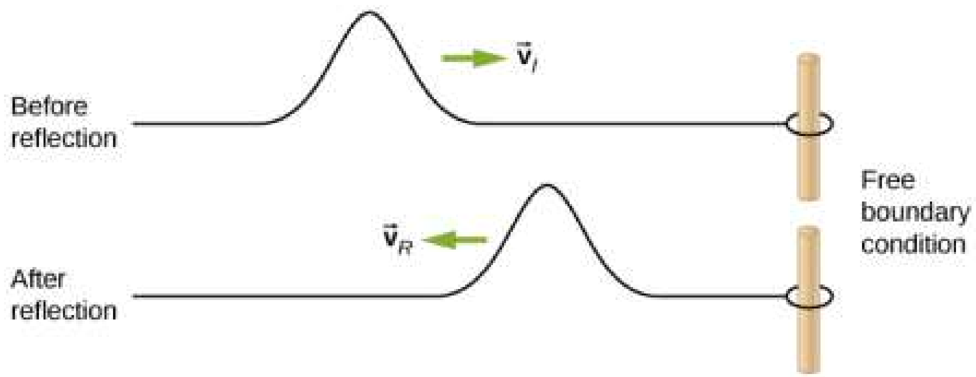
A **fixed boundary condition** exists when the medium at a boundary is fixed in place so it cannot move.



https://cnx.org/contents/5I39byUz@3.2:atV-gRHg@10/3-4-Interference-of-Waves

The reflected wave is reflected 180° out of phase with the incident wave.

A **free boundary condition** exists when the medium at the boundary is free to move.



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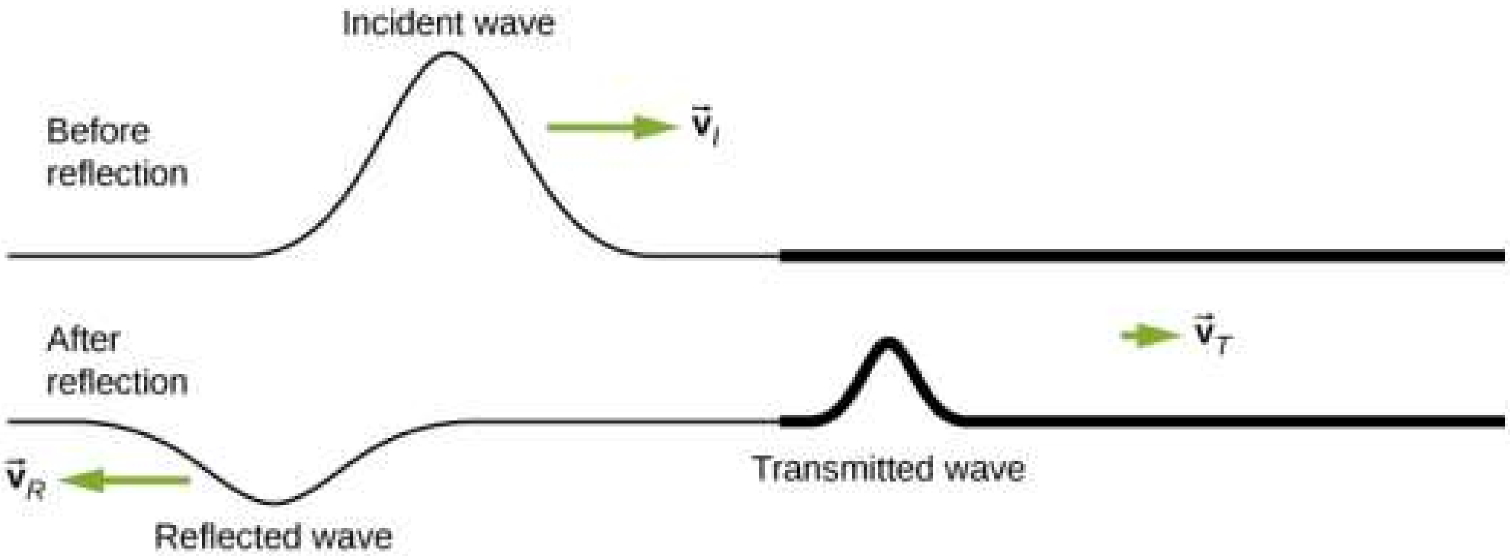
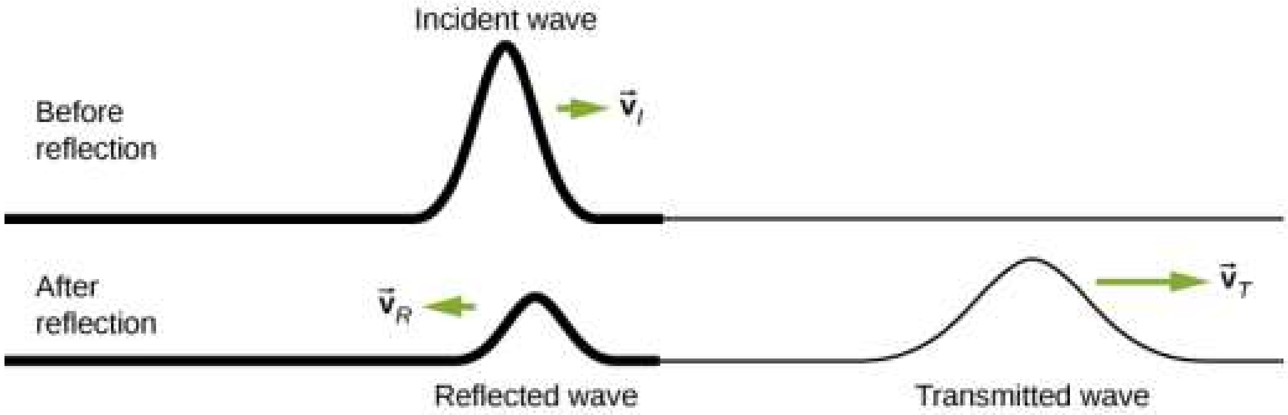
The reflected wave is in phase with respect to the incident wave.

In some situations, the boundary of the medium is neither fixed nor free.

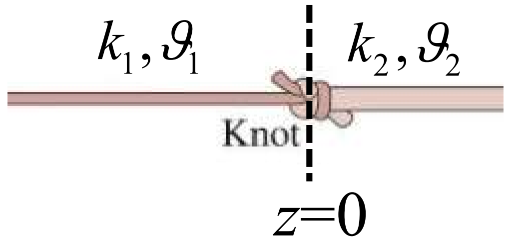


A low-linear mass density string is attached to a string of a higher linear mass density.



A high linear mass density string is attached to a string of a lower linear mass density.

Suppose that there is a knot between two strings at z=0 and the oscillations are at the same frequency. How are the reflected and transmitted waves in terms of the incident wave?



The incident wave,

$$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)}, \quad (z < 0)$$

The reflected wave,

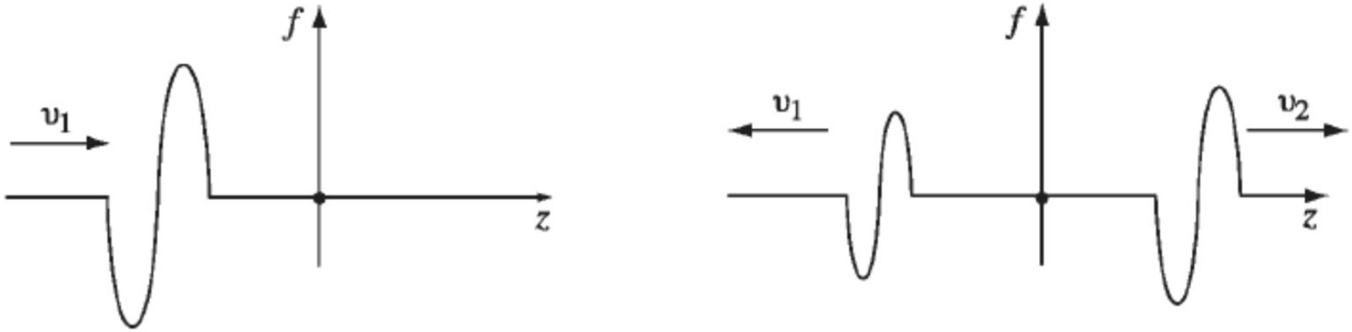
$$\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}, \quad (z < 0)$$

The transmitted wave,

$$\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)}, \quad (z > 0)$$

1st boundary condition: At the join z=0, the displacement just slightly to the left (z=0^-) must be equal the displacement to the right (z=0^+). Mathematically f(z,t) is continuous at z=0.

$$f(0^-, t) = f(0^+, t)$$



(a) Incident pulse

(b) Reflected and transmitted pulses

2nd boundary condition: If the knot itself is of negligible mass, then the derivative of f must also be continuous:

$$\left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+}$$

Since the imaginary part of the wave function differs from the real part by replacing cosine with sine, the complex wave function obeys the same rules:

$$\tilde{f}(0^-, t) = \tilde{f}(0^+, t), \qquad \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}$$

Remember that the wavelengths, wave numbers and wave velocities have the relationship:

$$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2}$$

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & \text{for } z < 0, \\ \tilde{A}_T e^{i(k_2 z - \omega t)}, & \text{for } z > 0. \end{cases}$$

$$\tilde{f}(0^-,t) = \tilde{f}(0^+,t), \quad \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}$$

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T \qquad k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T$$

$$\tilde{A}_R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\tilde{A}_I = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{A}_I$$

$$\tilde{A}_T = \left(\frac{2k_1}{k_1 + k_2}\right)\tilde{A}_I = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{A}_I$$

The real amplitudes and phases then are related by,

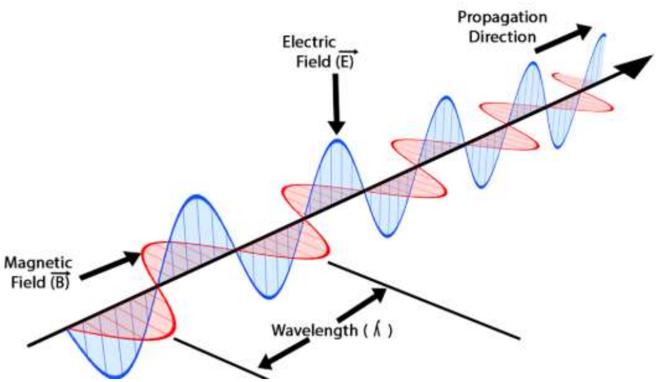
$$A_R e^{i\delta_R} = \left(\frac{\vartheta_2 - \vartheta_1}{\vartheta_2 + \vartheta_1}\right) A_I e^{i\delta_I} \qquad A_T e^{i\delta_T} = \left(\frac{2\vartheta_2}{\vartheta_2 + \vartheta_1}\right) A_I e^{i\delta_I}$$

(i) If the second string is lighter than the first ($\mu_2 < \mu_1$), all three waves have the same phase angle ($\delta_R = \delta_T = \delta_I$).
(ii) If the second string is heavier than the first ($\mu_2 > \mu_1$), the reflected wave is out of phase by 180°.
(iii) If the second string is infinitely massive, $A_R = A_I$ and $A_T = 0$.

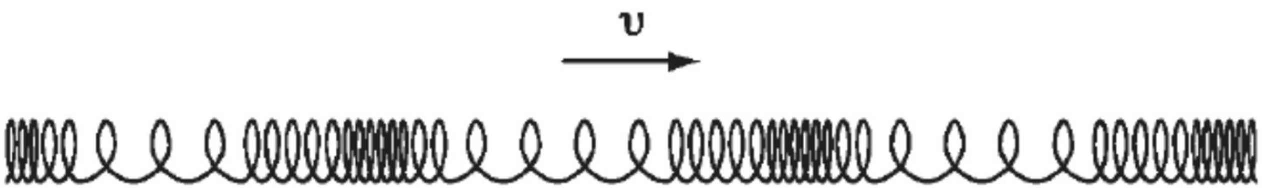
Write the real A_R and real A_T for the situations (i) and (ii).

9.1.4 Polarization

Transverse waves: Displacement from equilibrum is perpendicular to direction of propagation. For example, electromagnetic waves



Longitudinal waves: Displacement from equilibrum is along the direction of propagation. For example, sound waves



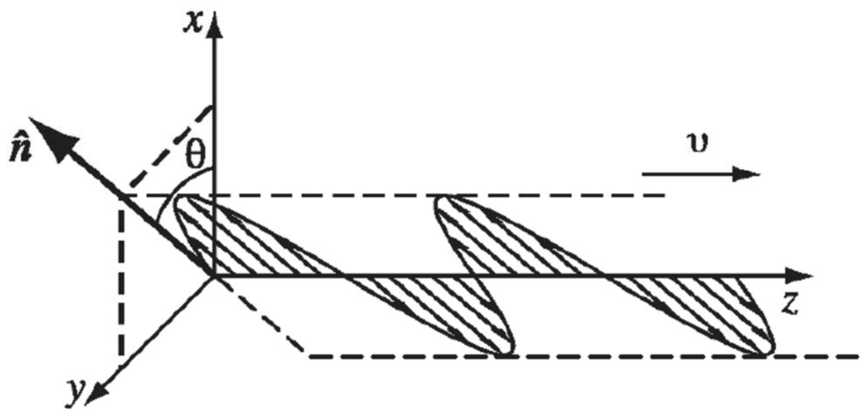
Transverse waves occur in two independent states of polarization: Vertical polarization (left-and-right) and horizontal polarization (up-and-down):

$$\tilde{\mathbf{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{x}} \qquad \tilde{\mathbf{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{y}}$$

General form: $\tilde{\mathbf{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{n}}$,

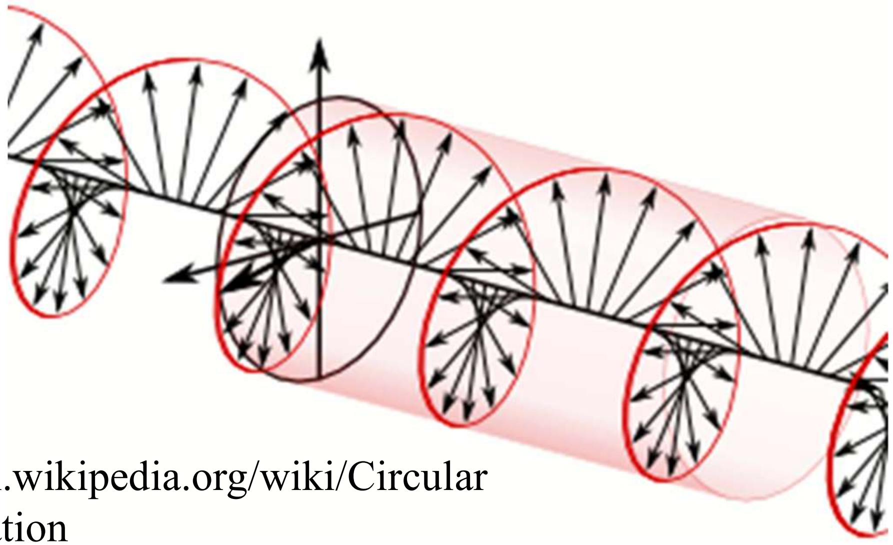
where $\hat{\mathbf{n}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$

$$\hat{n}.\hat{z} = 0$$



$$\tilde{\mathbf{f}}(z,t) = (\tilde{A}\cos\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{x}} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{y}}.$$

Circular Polarization



https://en.wikipedia.org/wiki/Circular
_polarization

- A circularly **polarized** wave can be in one of two possible states, **right circular polarization** in which the electric field vector rotates in a **right**-hand sense with respect to the direction of propagation, and **left circular polarization** in which the vector rotates in a left-hand sense.
- δ_x-δ_y=90° for circular polarized light

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