MTH 102 Calculus -II

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Topic-3-Applications of Integration

Areas between Curves

Applications of Integration

In this chapter we explore some of the applications of the definite integral by using it to compute **areas** between curves, **volumes** of solids, and the work done by a varying force.

2.1. Areas between Curves

In Chapter 1, we defined and calculated areas of regions that lie under the graphs of functions. Here we use integrals to find areas of regions that lie between the graphs of two functions.

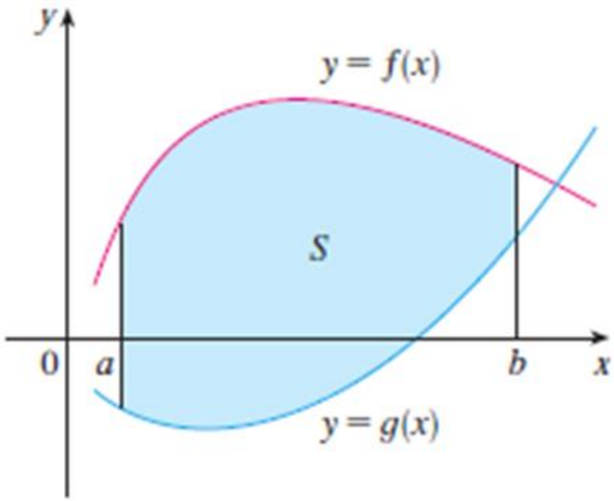

FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, g(x) \leqslant y \leqslant f(x)\}$

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$. (See Figure 1.)

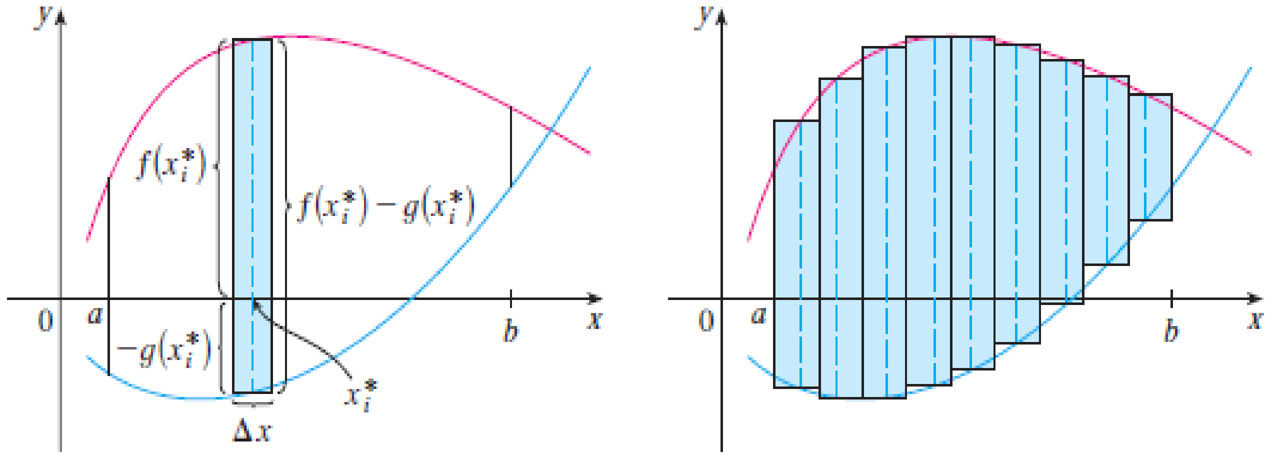

FIGURE 2 (a) Typical rectangle (b) Approximating rectangles

Just as we did for areas under curves in Section 1.1, we divide S into n strips of equal width and then we approximate the *ith* strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$. (See Figure 2. If we liked, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$.) The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

is therefore an approximation to what we intuitively think of as the area of S.

This approximation appears to become better and better as $n \to \infty$. Therefore we define the **area** of the region as the limiting value of the sum of the areas of these approximating rectangles.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x \qquad (1)$$

We recognize the limit in (1) as the definite integral of $f - g$. Therefore we have the following formula for area.

The area A of the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)] dx \qquad (2)$$

Note:

- Notice that in the special case where $g(x) = 0$, S is the region under the graph of f and our general definition of area (1) reduces to our previous definition (Definition 2 in Section 1.1).

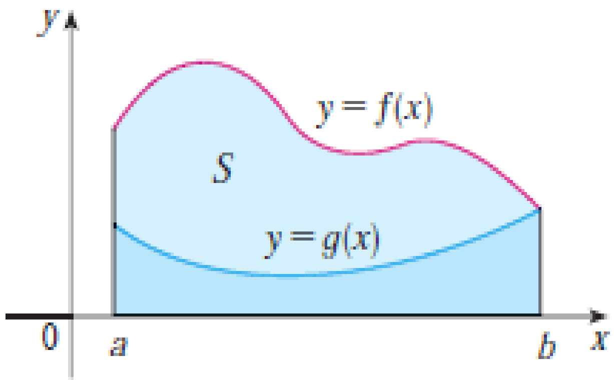

FIGURE 3

$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

- In the case where both f and g are positive, you can see from Figure 3 why (2) is true:

$$A = [area\ under\ y = f(x)] - [area\ under\ y = g(x)]$$

$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]dx$$

Example 1.

Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

SOLUTION The region is shown in Figure 4.

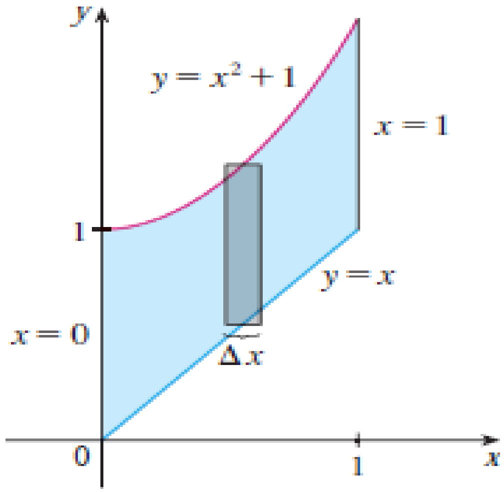

FIGURE 4

Steps for finding the area between two curves:

1. Sketch the two curves
2. Find the intersection points
3. Evaluate the integral

- In general, when we set up an integral for an area, it's helpful to sketch the region to identify the top curve y_T , the bottom curve y_B , and a typical approximating rectangle as in Figure 5.

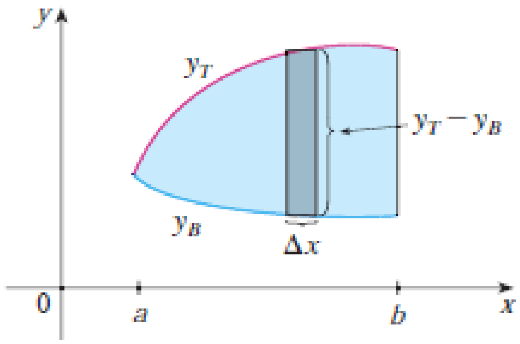

FIGURE 5

$$A = \int_a^b (y_T - y_B)dx$$

- Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y), x = g(y), y = c \; and \; y = d$ where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$(see Figure 11), then its area is

$$A = \int_c^d [f(y) - g(y)]dy$$

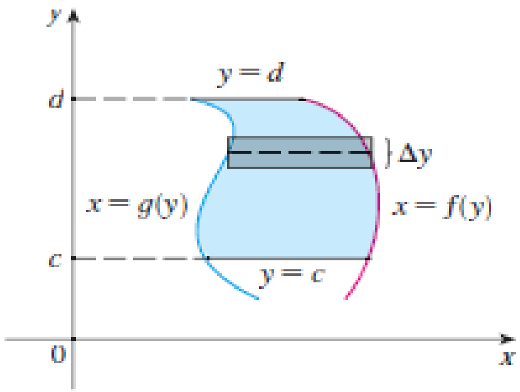

FIGURE 11

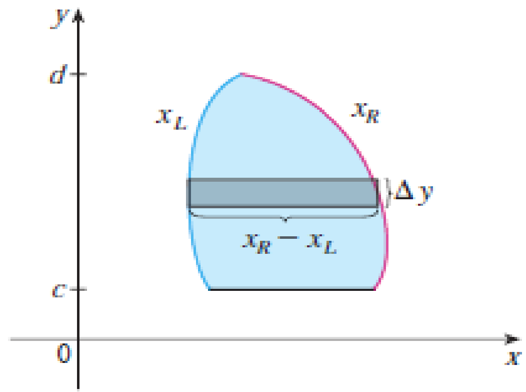

FIGURE 12

If we write x_R for the right boundary and x_L for the left boundary, then, as Figure 12 illustrates, we have

$$A = \int_c^d [x_R - x_L] \, dy$$

Example 5 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution

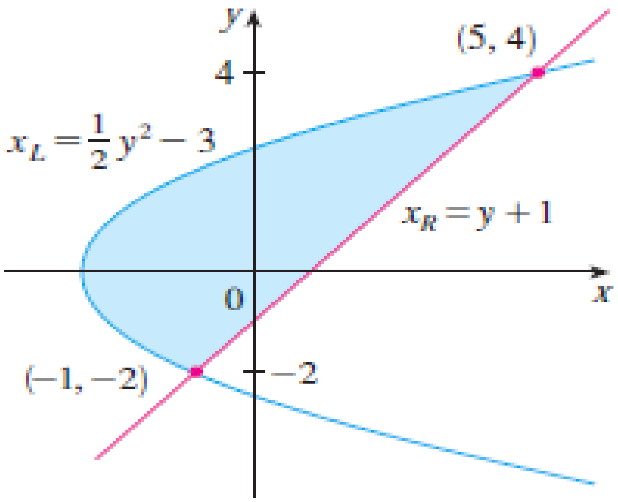

FIGURE 13

By solving the two equations we find that the points of intersection are (-1,-2) and (5,4) . We solve the equation of the parabola for x and notice from Figure13 that the left and right boundary curves are

$$x_L = \frac{1}{2}y^2 - 3 \; and \; x_R = y + 1$$

We must integrate between the appropriate y-values, $y = -2 \; and \; y = 4$. Thus

$$A = \int_{-2}^{4} [(y + 1\,) - (\frac{1}{2}y^2 - 3)]dy = 18$$