

MTH 102 Calculus -II

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Topic-6-Improper Integrals

3.1 Improper Integrals

Up to now, definite integrals have been required to have two properties.

1. The domain of the integration $[a, b]$ finite,
2. The range of the integrand does not have an infinite discontinuity.

In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f (integrand) has an infinite discontinuity in $[a, b]$. In either case the integral is called an **improper integral**.

Type 1: Infinite Intervals

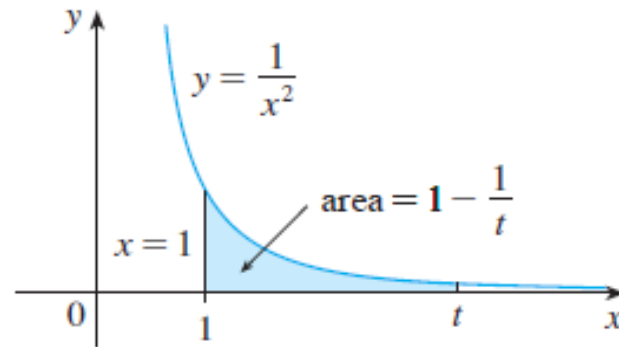


FIGURE 1

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x -axis, and to the right of the line $x = 1$. You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of that lies to the left of the line $x = t$ (shaded in Figure 1) is

$$A(t) = \int_1^t \frac{1}{x^2} dx = \left. \frac{-1}{x} \right|_1^t = 1 - \frac{1}{t}$$

Notice that $A(t) < 1$ no matter how large t is chosen.

We also observe that

$$\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} \left(1 - \frac{1}{t} \right) = 1$$

The area of the shaded region approaches 1 as $t \rightarrow \infty$ (see Figure 2), so we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx = 1$$

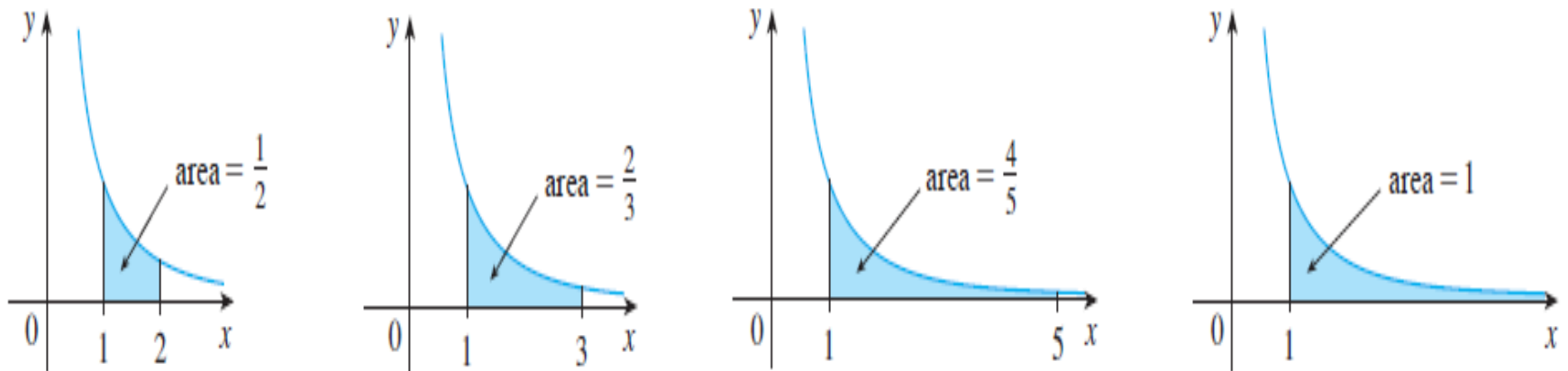


FIGURE 2

Using this example as a guide, we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x) dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x) dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x) dx$ and $\int_{-\infty}^b f(x) dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x) dx$ and $\int_{-\infty}^a f(x) dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$$

In part (c) any real number a can be used (see Exercise 74).

Now let's solve some examples!!!

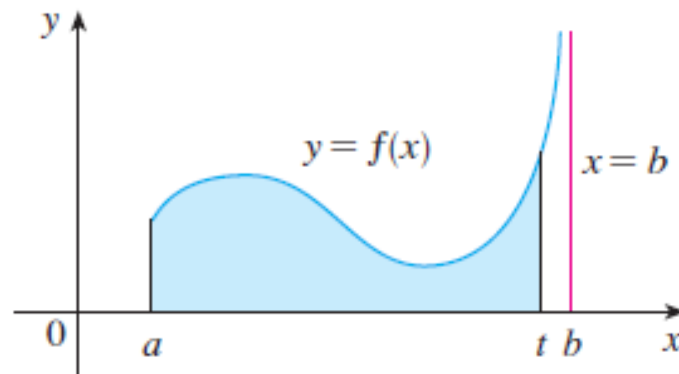
Type 2: Discontinuous Integrands

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b . Let S be the unbounded region under the graph of f and above the x - axis between a and b . (For Type 1 integrals, the regions extended indefinitely in a horizontal direction. Here the region is infinite in a vertical direction.) The area of the part of S between a and t (the shaded region in Figure) is

$$A = \int_a^t f(x) dx$$

If it happens that $A(t)$ approaches a definite number A as $t \rightarrow b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$



3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a , then

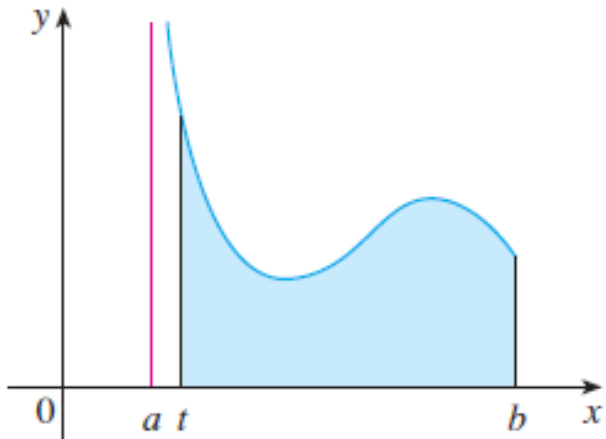
$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

if this limit exists (as a finite number).

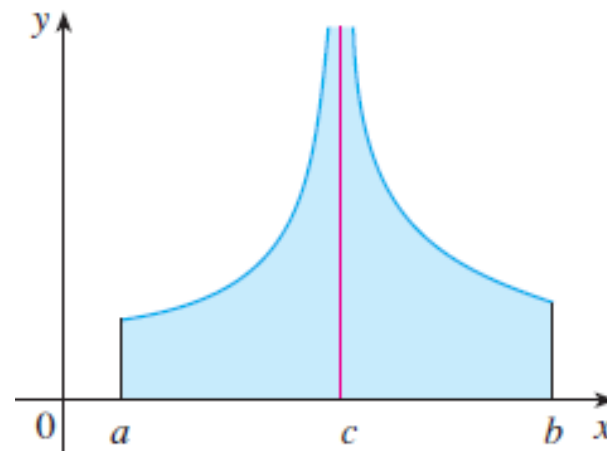
The improper integral $\int_a^b f(x) dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c , where $a < c < b$, and both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ are convergent, then we define

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$



These figures illustrates the definition



Now let's solve some examples!!!

Definition of an Improper Integral of Type 3

If an integral has the properties of both Type 1 and 2, then it is said that the integral is improper integral of Type 3.