References

- James Stewart, Calculus, Thomson, 2002
- George B. Thomas, Thomas' Calculus, Pearson, 2005
- Dr.Karakoc's Calculus Lecture Notes

INTEGRALS

This chapter starts with the area problems and uses them to formulate the idea of a definite integral, which is the basic concept of integral calculus.

1.1 The Area Problem

We begin by attempting to solve the area problem:

Find the area of the region that lies under the curve

$$y = f(x)$$

from a to b.

This means that , illustrated in Figure 1, is bounded by the graph of a continuous function f [$where\ f(x) \geq 0$], $the\ vertical\ lines\ x = a\ and\ x = b, and\ the\ x - axis$.

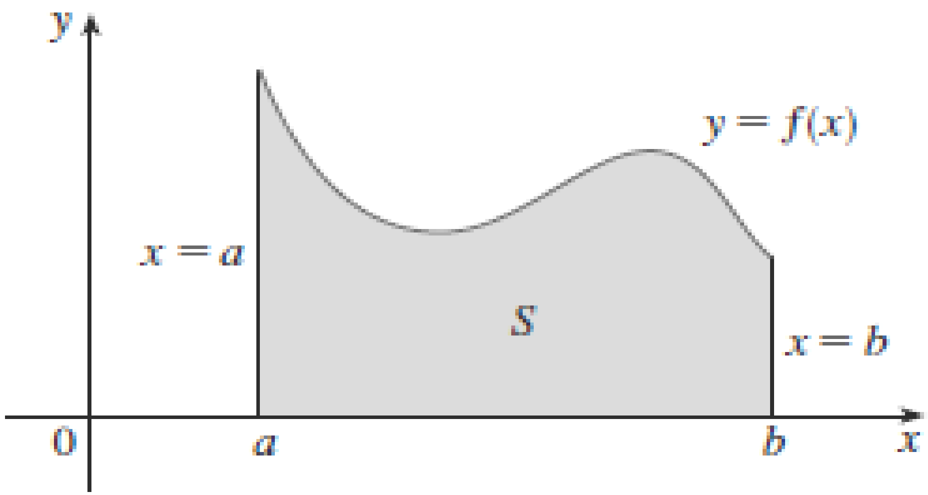


Figure 1

In trying to solve the area problem we have to ask ourselves: What is the meaning of the word **area?** This question is easy to answer for regions with **straight sides**.

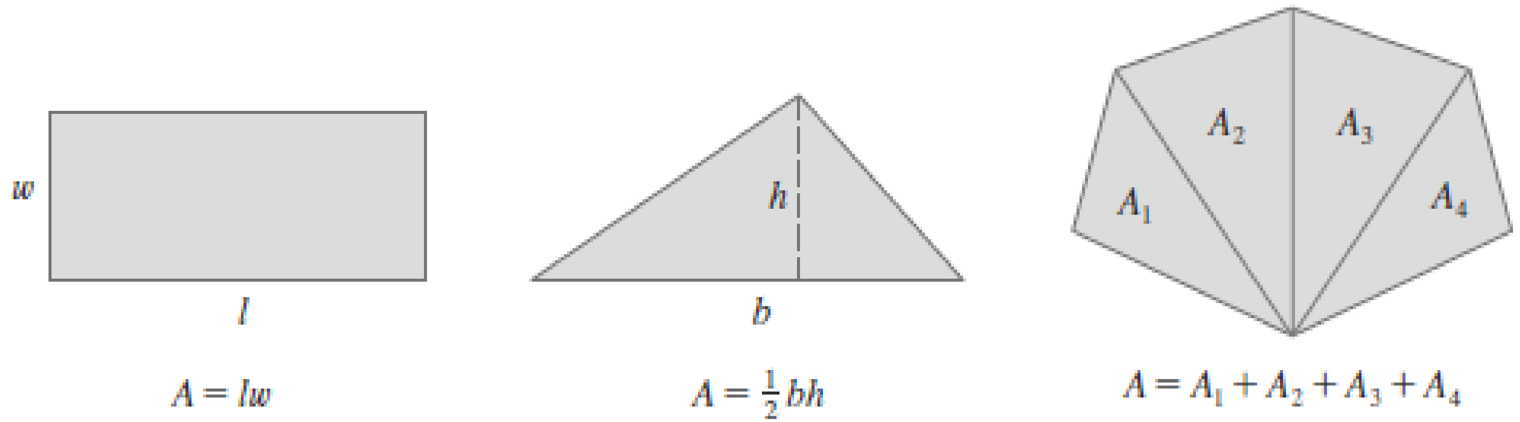


Figure 2

- For a rectangle, the area is defined as the product of the length and the width.

- The area of a triangle is half the base times the height.

- The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

However, it isn’t so easy to find the area of a region with **curved sides**. We all have an intuitive idea of what the area of a region is.
But part of the area problem is to make this intuitive idea precise by giving an **exact definition** of area.

We first approximate the region by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

- The following example illustrates the procedure.

Example-1

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

(the parabolic region S illustrated in Figure 3).

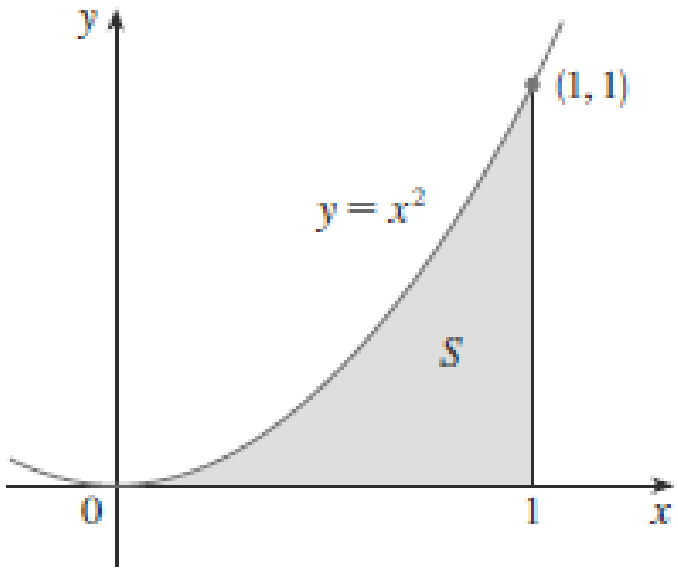
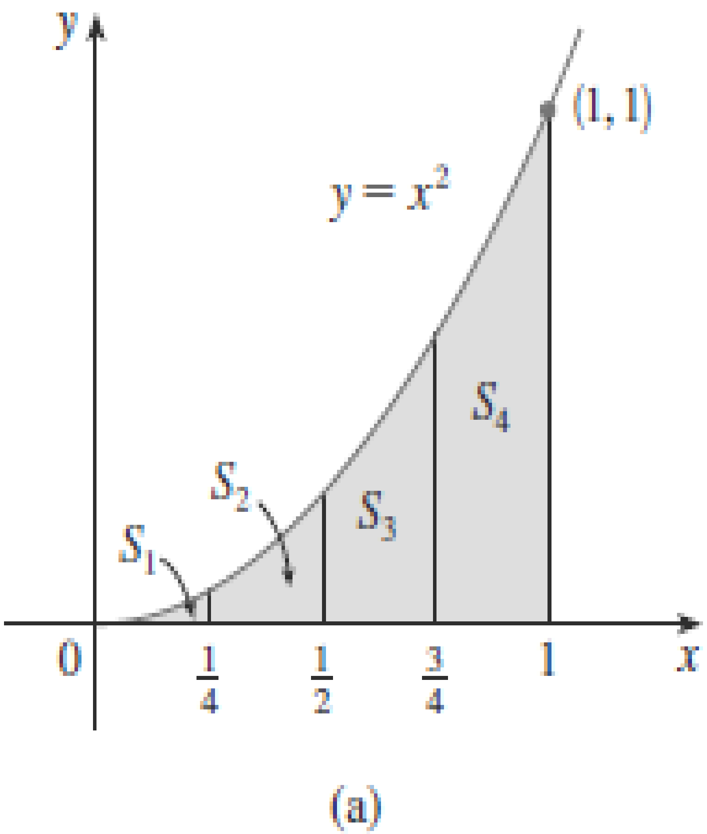


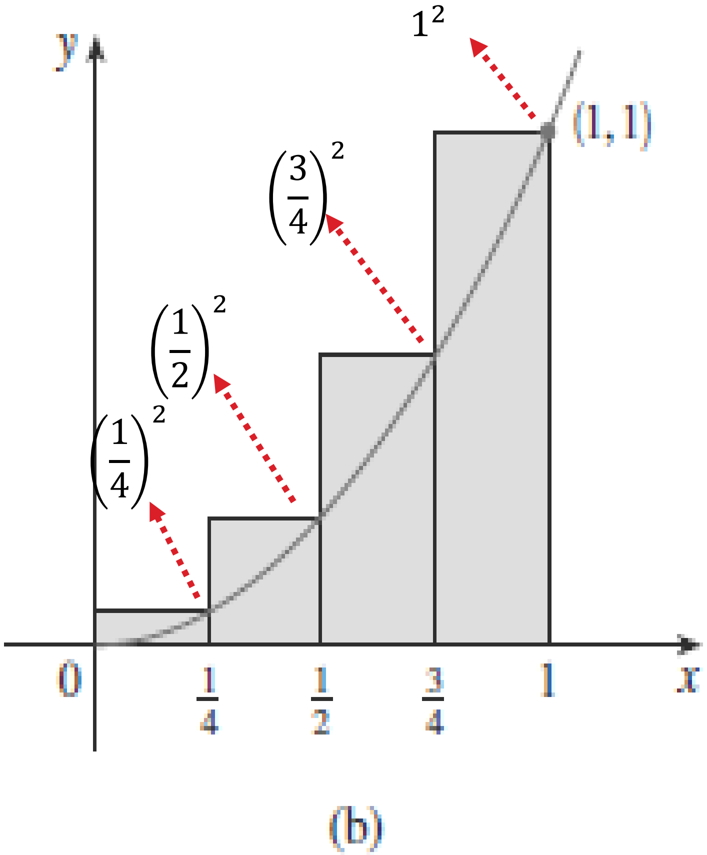
Figure 3

Solution

We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that.



y
x
0
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
(1, 1)
$y = x^2$
S_1
S_2
S_3
S_4
(a)



y
x
0
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
(1, 1)
1^2
$\left(\frac{3}{4}\right)^2$
$\left(\frac{1}{2}\right)^2$
$\left(\frac{1}{4}\right)^2$
(b)

Figure 4

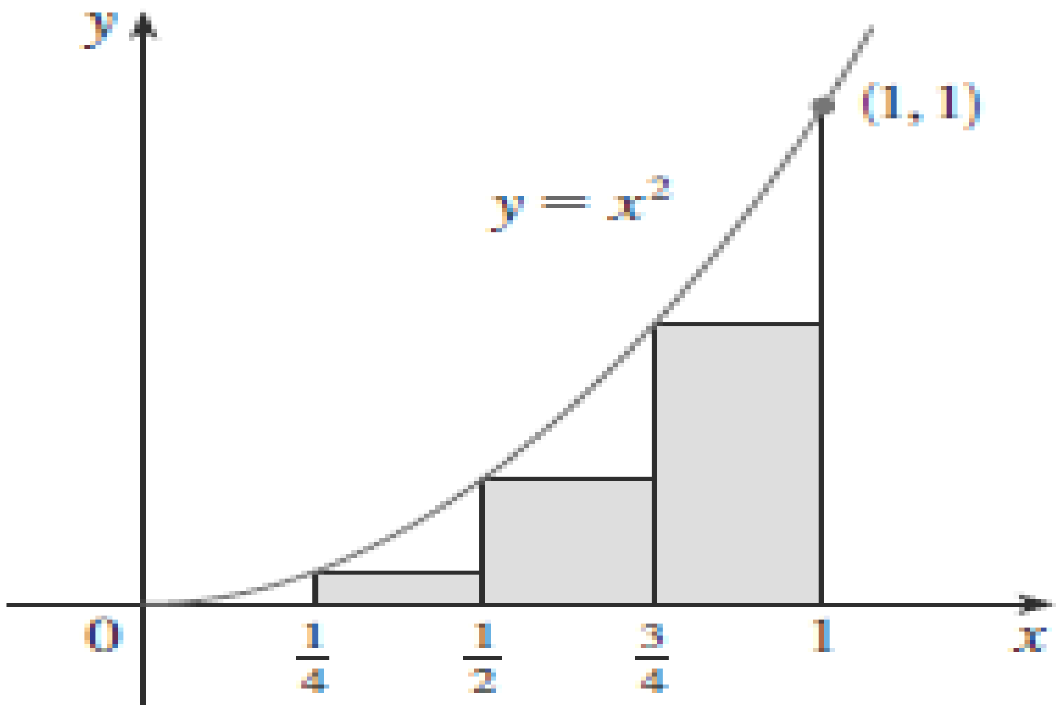


Figure 5

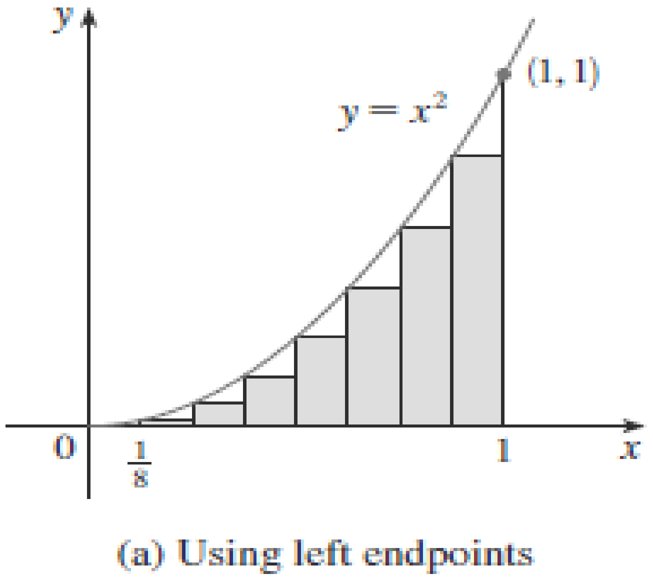

y
$y = x^2$
(1, 1)
0
$\frac{1}{8}$
1
x
(a) Using left endpoints
y
(1, 1)
0
$\frac{1}{8}$
1
x
(b) Using right endpoints

Figure 6

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

From the values in the table in Example 1, it looks as if R_n is approaching $\frac{1}{3}$ as n increases.

We confirm this in the next example.

Example-2

For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is

$$\lim_{n \to \infty} R_n = \frac{1}{3}$$

Solution

R_n is the sum of the areas of the n rectangles in Figure 7.

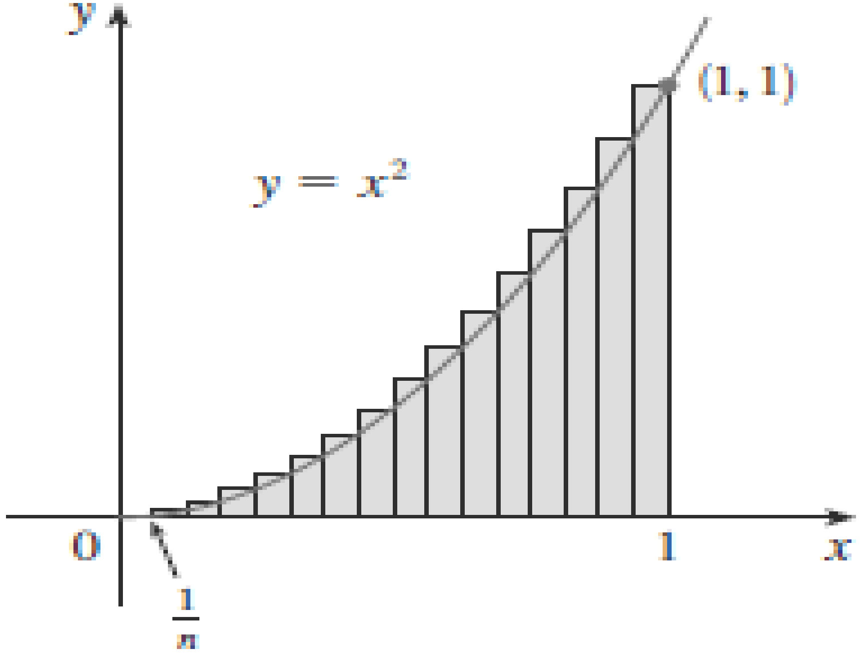


Figure 7

From Figures 8 and 9 it appears that, as n increases, both L_n and R_n become better and better approximations to the area of S.

Therefore, we define the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}$$

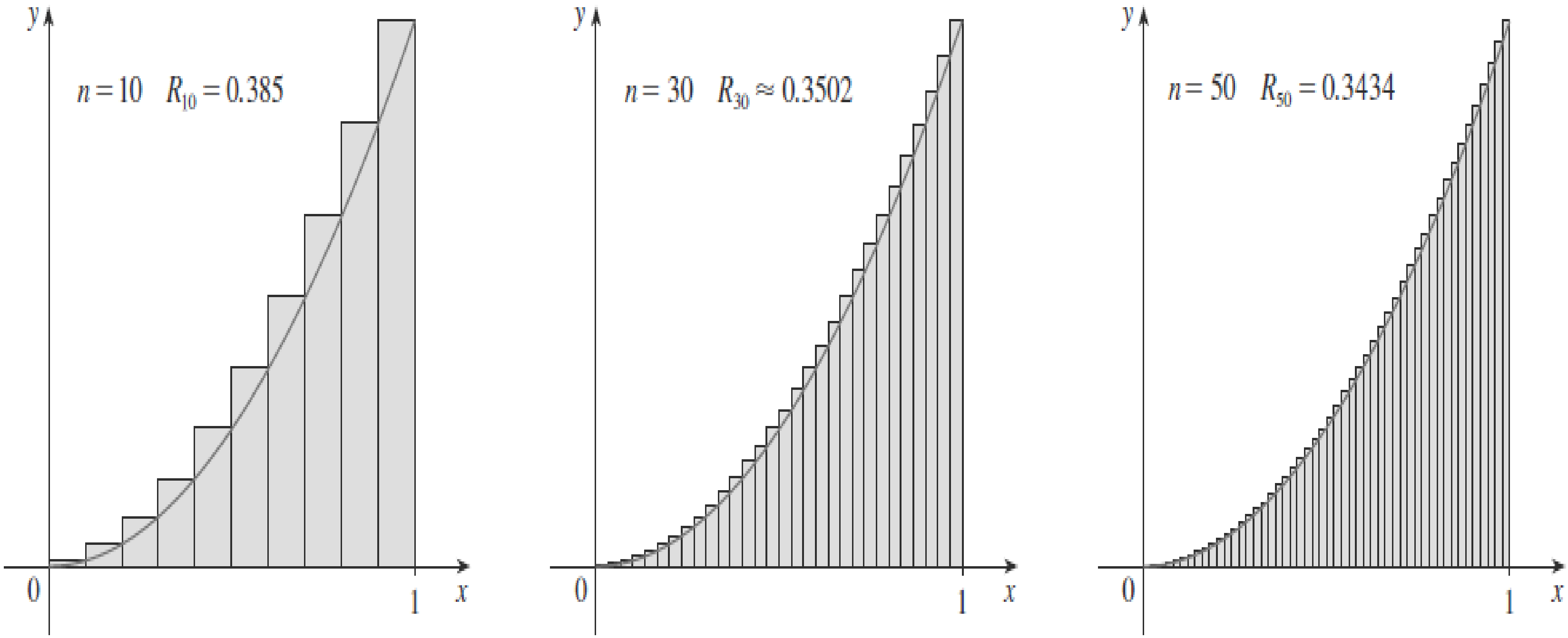

y
$n = 10 \quad R_{10} = 0.385$
0
1
x
y
$n = 30 \quad R_{30} \approx 0.3502$
0
1
x
y
$n = 50 \quad R_{50} = 0.3434$
0
1
x

FIGURE 8

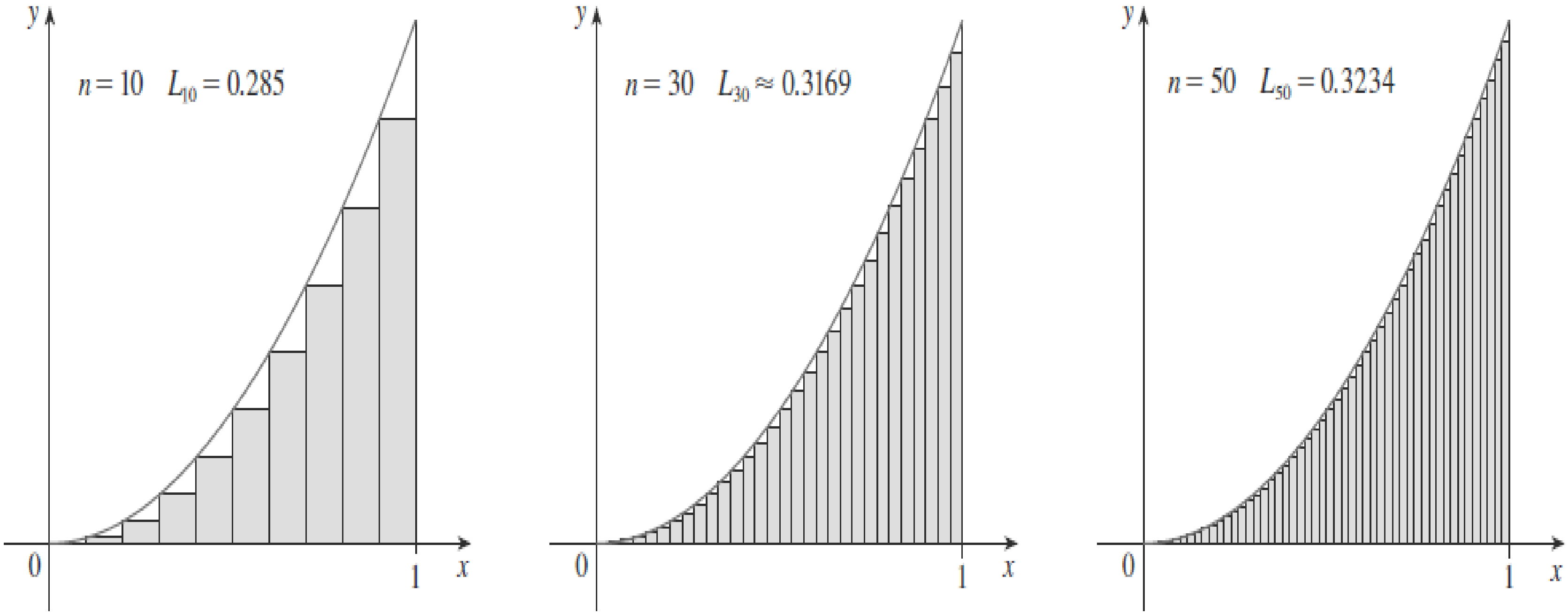


FIGURE 9

The area is the number that is smaller than all upper sums and larger than all lower sums

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \dots, S_n$ of equal width as in Figure 10.

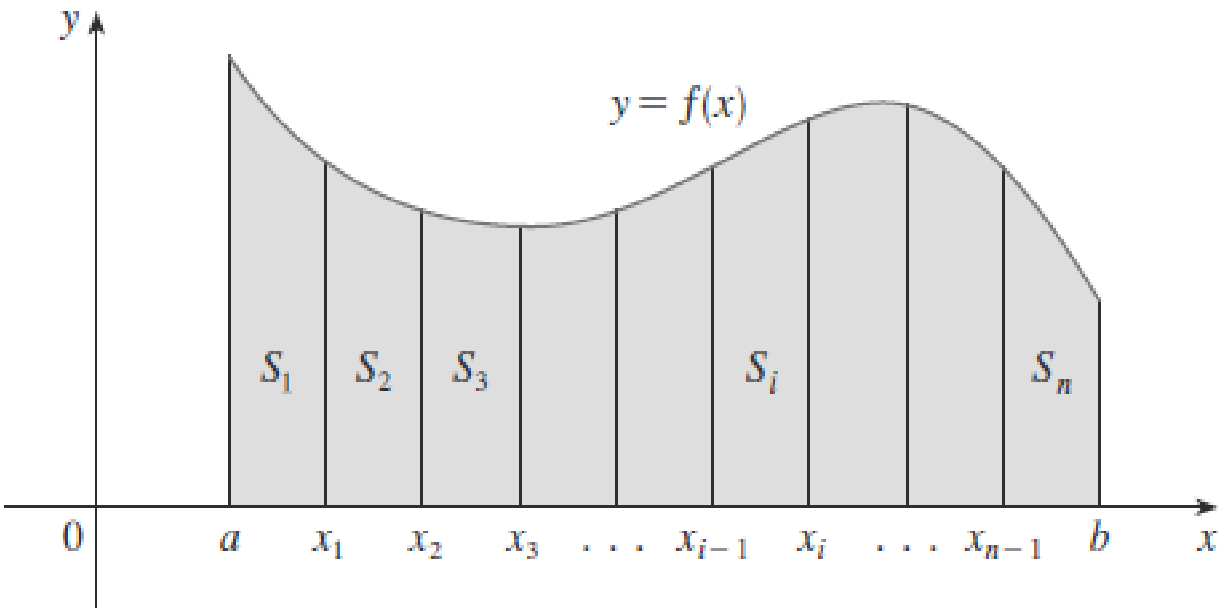


Figure 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b-a}{n} .$$

These strips divide the interval [a, b] into n subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$.

The right endpoints of the subintervals are

$x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ x_3 = a + 3\Delta x, \ldots$

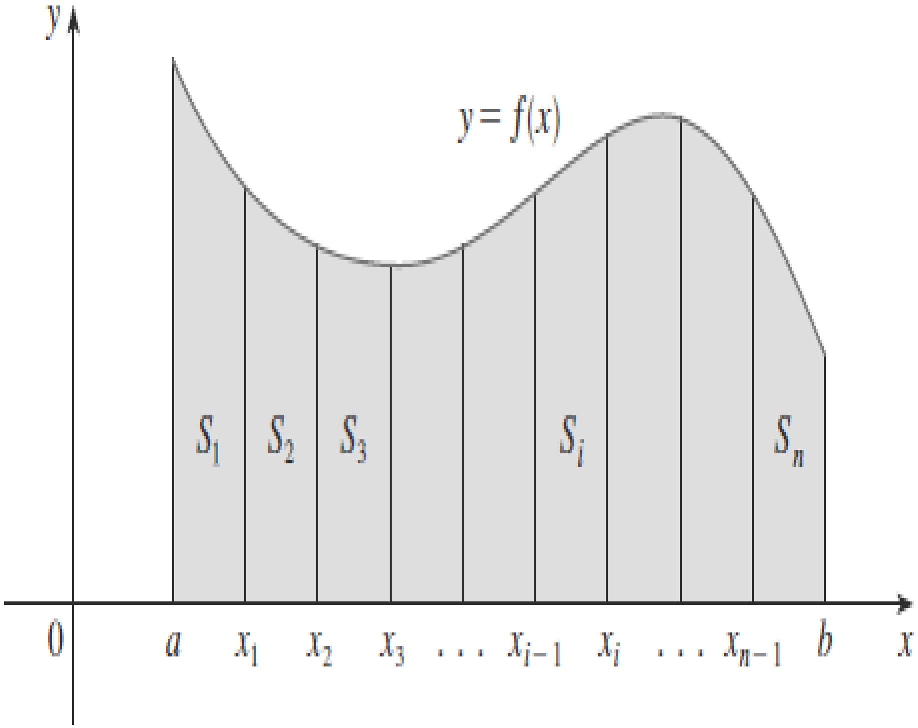


FIGURE 10

Let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 11).

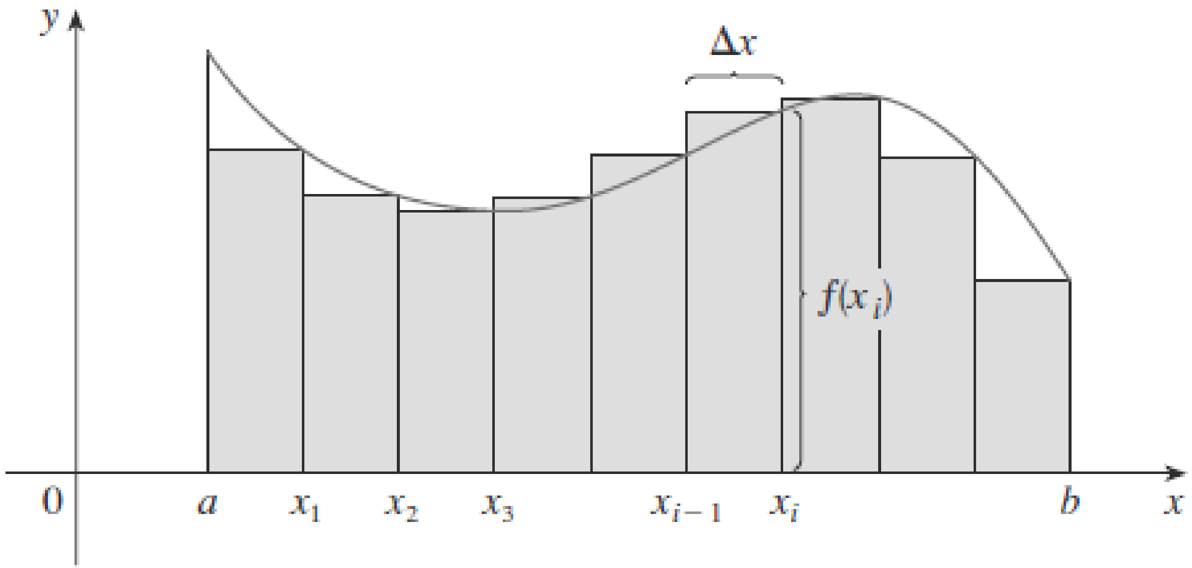


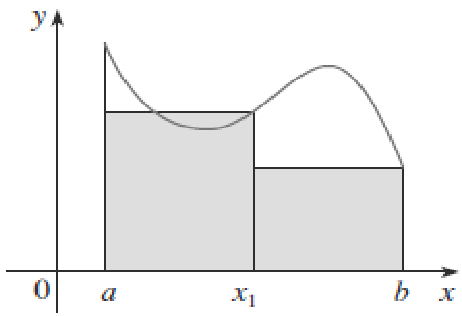
Figure 11

Then the area of the ith rectangle is $f(x_i)\Delta x$.

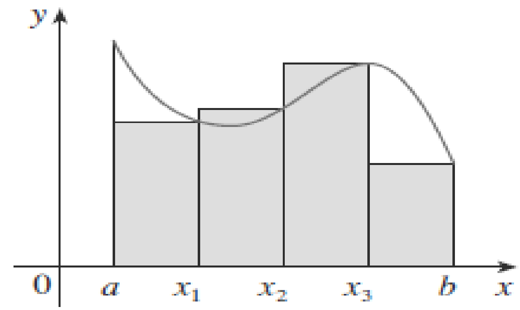
What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

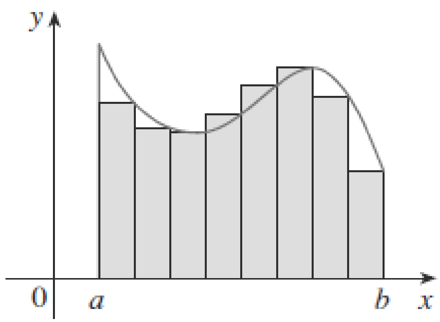
Figure 12 shows this approximation for n 2, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.



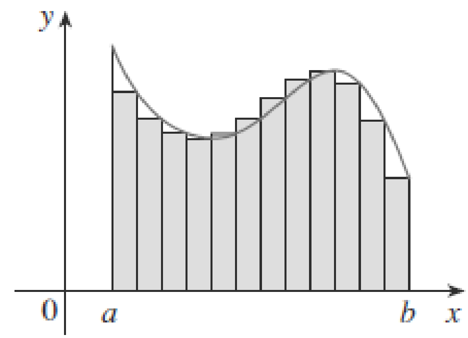
(a) $n = 2$



(b) $n = 4$



(c) $n = 8$



(d) $n = 12$

Figure 12

Therefore, we define the area A of the region S in the following way.

Definition 1. The **area** A of the region S that lies under the graph of the continuous function is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [\, f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] \text{ (1)}$$

It can be proved that the limit in Definition 1 always exists, since we are assuming that is continuous.

It can also be shown that we get the same value if we use left endpoints:

$$A = \lim_{n\to\infty} L_n$$
$$= \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x] \quad (2)$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at ***any*** number x_i^* in the ith subinterval $[x_{i-1}, x_i]$.

We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ **the sample points.**

Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints.

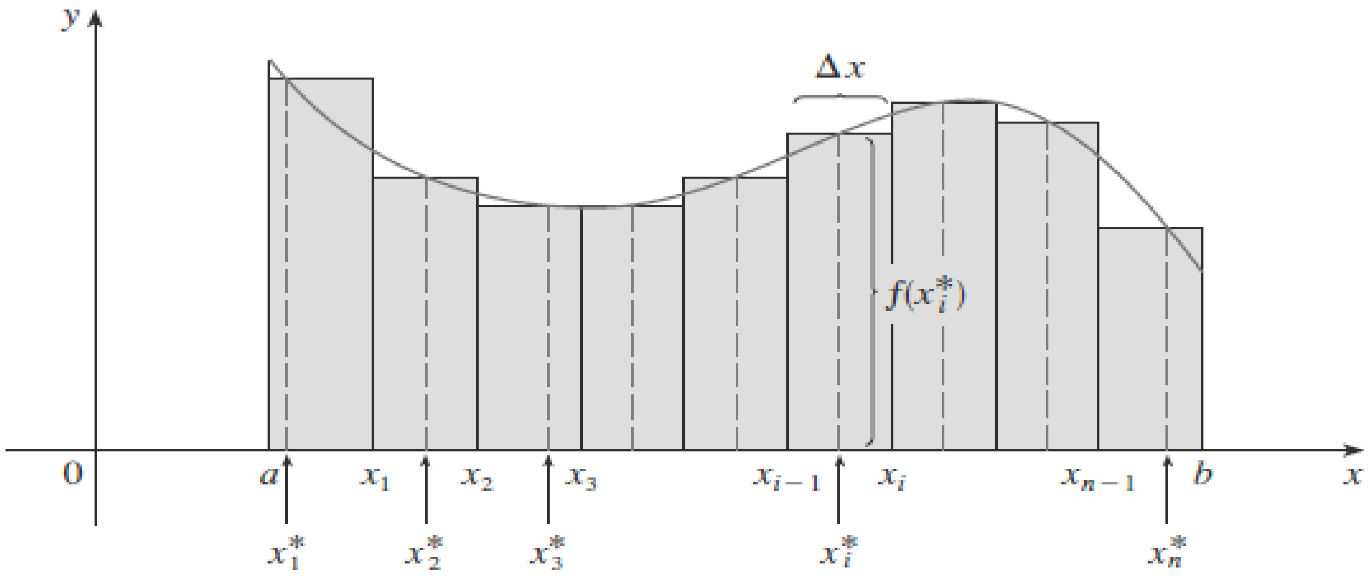


Figure 13

So a more general expression for the area of S is

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] \quad (3)$$

We often use sigma notation to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

So the expressions for area in Equations 1, 2, and 3 can be written as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Example –3

Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Solution

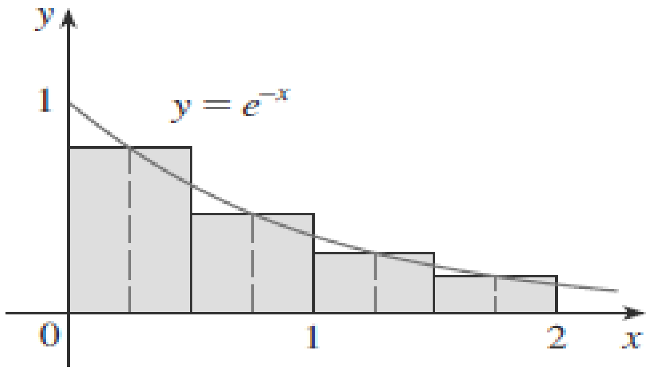


FIGURE 14

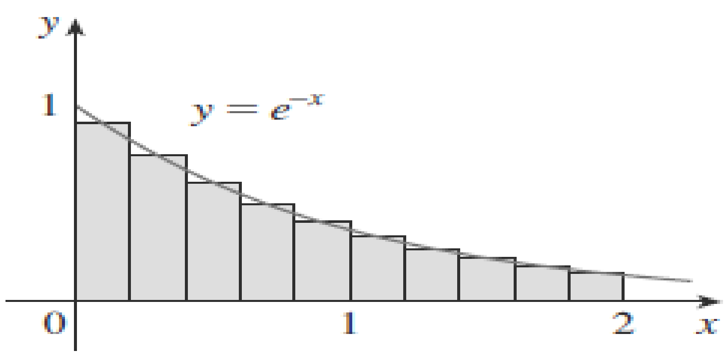


FIGURE 15

1.2 The Definite Integral

We saw in Section 1.1 that a limit of the form

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}[\, f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] \quad (1)$$

arises when we compute an area.

We also point out that limits of the form (1) also arise in finding

- lengths of curves,
- volumes of solids,
- centers of mass,
- force due to water pressure,

and work, as well as other quantities.

We therefore give this type of limit a special name and notation.

Definition 2 (Definite Integral)

If f is a continuous function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0(= a)$, $x_1, x_2, \dots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of $\boldsymbol{f}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad (2)$$

Because we have assumed that f is continuous, it can be proved that the limit in Definition 2 always exists and gives the same value no matter how we choose the sample points x_i^*.

If we take the sample points to be **right endpoints**, then $x_i^* = x_i$ and the definition of an integral becomes

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad (3)$$

If we choose the sample points to be **left endpoints**, then $x_i^* = x_{i-1}$ and the definition becomes

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Alternatively, we could choose x_i^* to be the midpoint of the subinterval or any other number between x_{i-1} and x_i.

Although most of the functions that we encounter are continuous, the limit in Definition 2 also exists if has a finite number of removable or jump discontinuities (but not infinite discontinuities.) So we can also define the definite integral for such functions.

Note 1

The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is **the lower limit** and is b **the upper limit**. The symbol dx has no official meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The procedure of calculating an integral is called **integration**.

Note 2

The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x . In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

Note 3

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). We know that if f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1). By comparing Definition 2 with the definition of area in Section 1.1, we see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b.

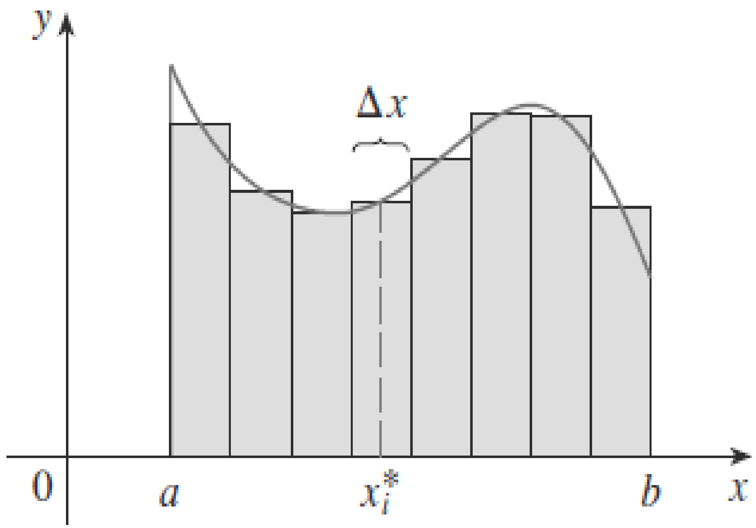


FIGURE 1

If $f(x) \geqslant 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

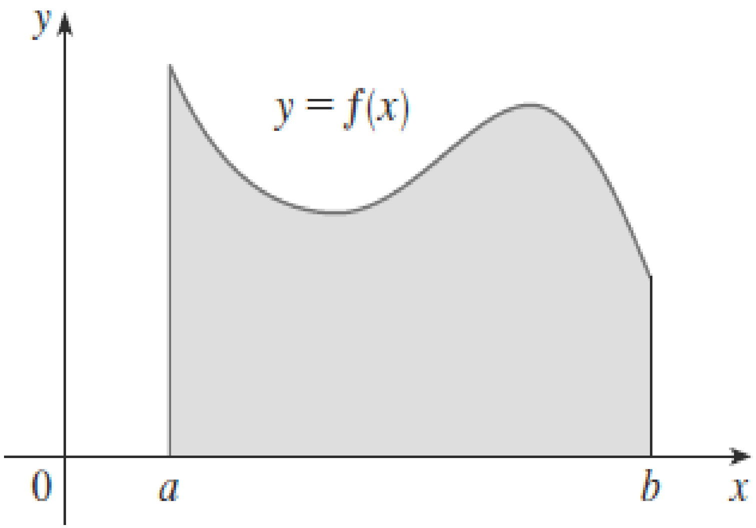


FIGURE 2

If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

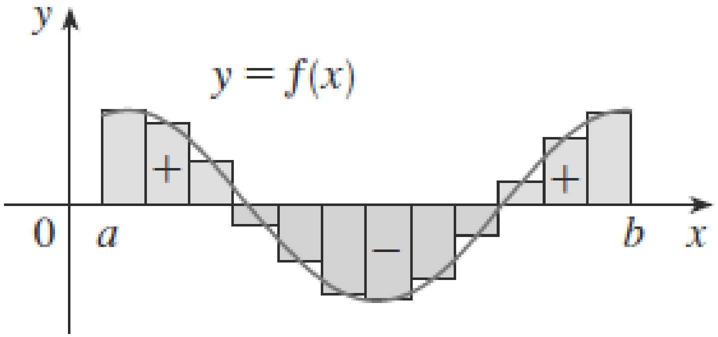


FIGURE 3
$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis.

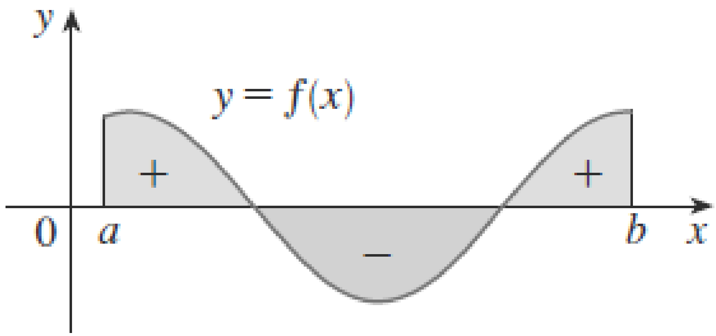


FIGURE 4

$\int_a^b f(x)\,dx$ is the net area

When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

Note 4

In the spirit of the precise definition of the limit of a function, we can write the precise meaning of the limit that defines the integral in Definition 2 as follows:

For every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

Note 5

Although we have defined $\int_a^b f(x)\,dx$ by dividing $[a,b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width. And there are methods for numerical integration that take advantage of unequal subintervals.

If the subinterval widths are Δx_1, Δx_2, ..., Δx_n we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Example-1

Express

$$\lim_{n\to\infty} \sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$$

as an integral on the interval $[0,\pi]$.

Evaluating Integrals

When we use the definition to evaluate a definite integral, we need to know how to work with sums.

The following three equations give formulas for sums of powers of positive integers.

Equation 4 may be familiar to you from a course in algebra. Equations 5 and 6 were discussed in Section 1.1.

$$\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2} \qquad (4)$$

$$\sum_{i=1}^{n} i^2 = \frac{n\,(n+1)(2n+1)}{6} \qquad (5)$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n\,(n+1)}{2}\right]^2 \qquad (6)$$

The remaining formulas are simple rules for working with sigma notation:

$$\sum_{i=1}^{n} c = nc \qquad (7)$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \qquad (8)$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \qquad (9)$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \qquad (10)$$

Example-2

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right endpoints and $a = 0, \mathrm{b} = 3$ and $\mathrm{n} = 6$.

(b) Evaluate $\int_0^3 x^3 - 6x\, dx$.

Solution

(a)

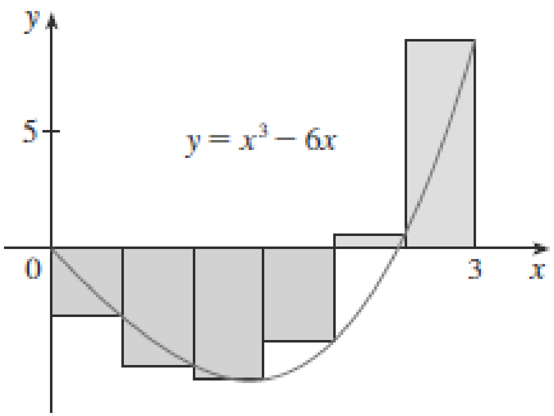


FIGURE 5

Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the **above the x-axis minus the sum of the areas of below the x-axis** in Figure 5.

(b)

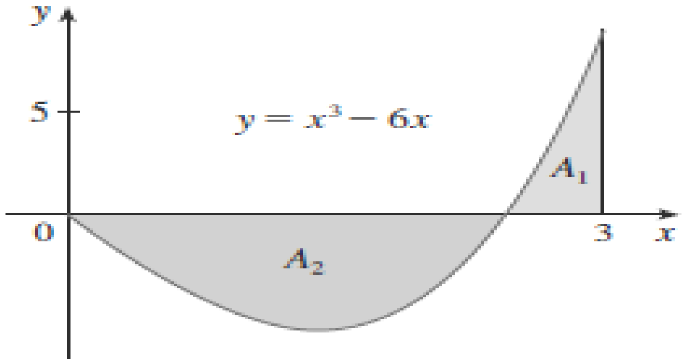


FIGURE 6
$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$

This integral can't be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas , $A_1 - A_2$ where A_1 and A_2 are shown in Figure 6.

Figure 7 illustrates the calculation by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$. The values in the table show the Riemann sums approaching the exact value of the integral, -6.75, as n → ∞.

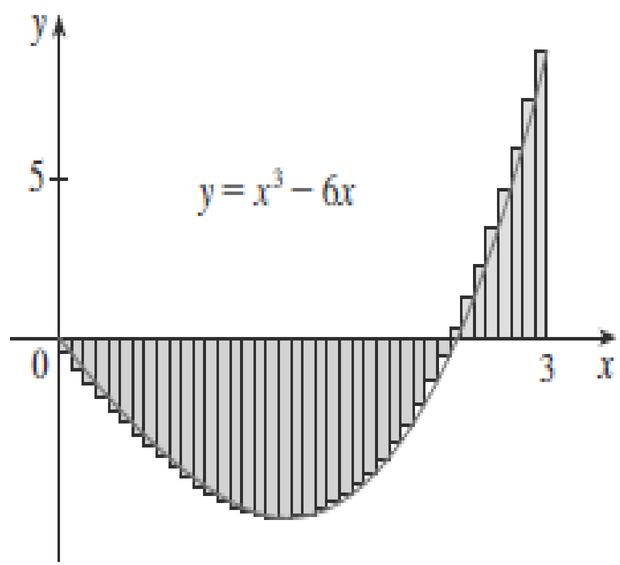


FIGURE 7
$R_{40} \approx -6.3998$

n	R_n
40	−6.3998
100	−6.6130
500	−6.7229
1000	−6.7365
5000	−6.7473

Example-3

Set up an expression for $\int_1^3 e^x \, dx$ as a limit of sums.

Solution

Because $f(x) = e^x$ is positive, the integral in Example 3 represents the area shown in Figure 8.

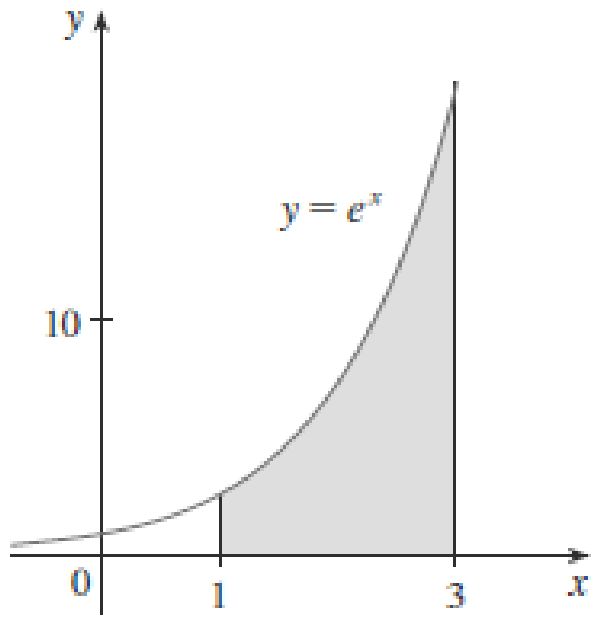


FIGURE 8

Example-4

Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2}\; dx$ (b) $\int_0^3 (x-1)\, dx$

Solution

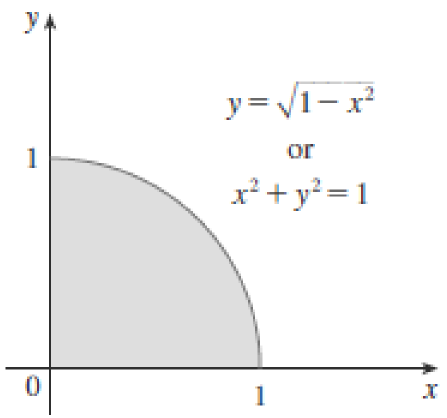


FIGURE 9

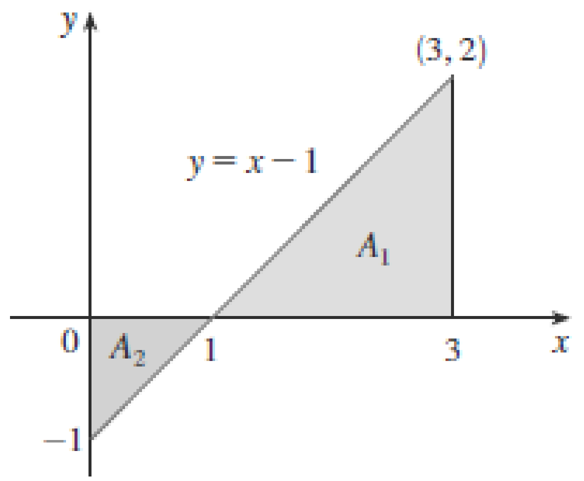


Figure 10

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the ith subinterval because it is convenient for computing the limit. But if the purpose is to find an approximation to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\overline{x_i}$. Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x = \Delta x[f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i) = midpoint\ of\ [x_{i-1}, x_i]$$

Example-5

Use the Midpoint Rule with n = 5 to approximate

$$\int_1^2 \frac{1}{x}\,dx$$

Solution

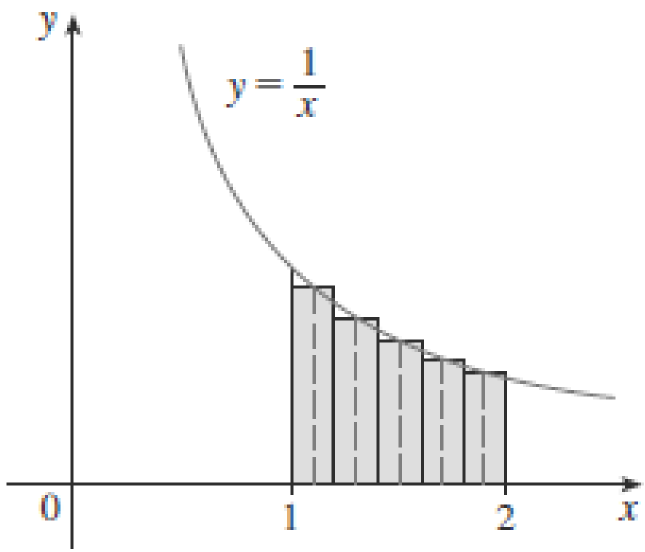


FIGURE 11

If we apply the Midpoint Rule to the integral in Example 2, we get the picture in Figure 12. The approximation $M_{40} \approx -6.7563$ is much closer to the true value -6.75 than the right endpoint approximation, $R_{40} \approx -6.3998$, shown in Figure 7.

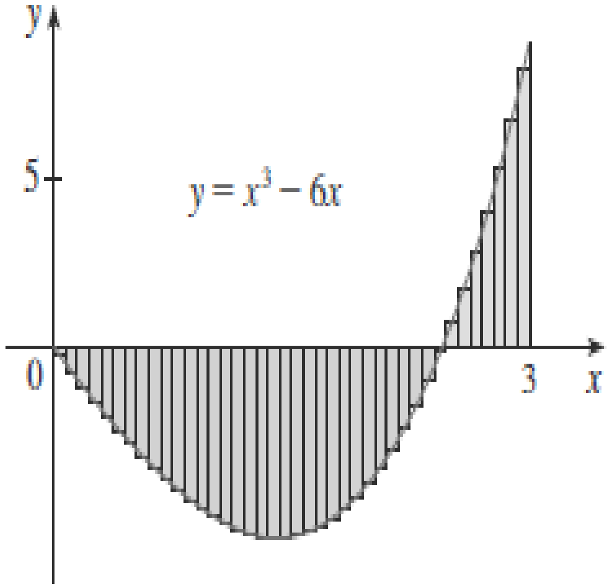


FIGURE 12
$M_{40} \approx -6.7563$

Properties of the Definite Integrals

When we defined the definite integral $\int_a^b f(x)dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we reverse a and b, then Δx changes from $(b-a)/n$ to $(a-b)/n$. Therefore

$$\boxed{\int_b^a f(x)dx = -\int_a^b f(x)dx}$$

If $a = b$, then $\Delta x = 0$ and so

$$\boxed{\int_a^a f(x)dx = 0}$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that f and g are continuous functions.

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Example-6

Use the properties of integrals to evaluate

$$\int_0^1 (4 + 3x^2)dx.$$

The next property tells us how to combine integrals of the same function over adjacent intervals:

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This is not easy to prove in general, but for the case where $f(x) \geqslant 0$ and $a < c < b$ Property 5 can be seen from the geometric interpretation in Figure 15: The area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

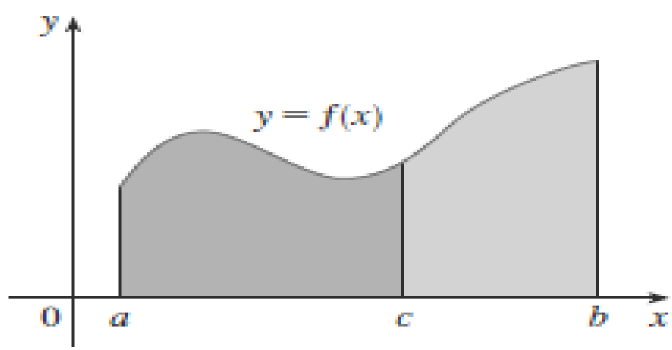


FIGURE 15

Example-7

If it is known that $\int_0^{10} f(x)dx = 17$ and $\int_0^{8} f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

Notice that Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The following properties, in which we compare sizes of functions and sizes of integrals, are true only if $a \leqslant b$.

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

For Property 8, note that if f is continuous we could take m and M to be the absolute minimum and maximum values of f on the inteval $[a, b]$.

Example-8

Use Property 8 to estimate $\int_0^1 e^{-x^2}\,dx$.

Solution

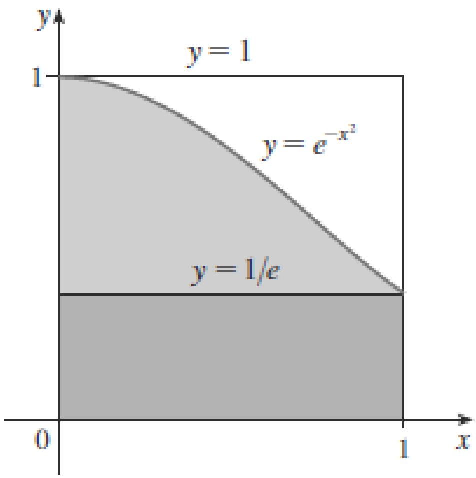


FIGURE 17

The result of Example 8 is illustrated in Figure 17. The integral is greater than the area of the lower rectangle and less than the area of the square.