Review of the Definite Integral

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leqslant x \leqslant b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$\boxed{1} \qquad \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

$$\boxed{2} \qquad \int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

In the special case where $f(x) \geqslant 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$ from a to b.

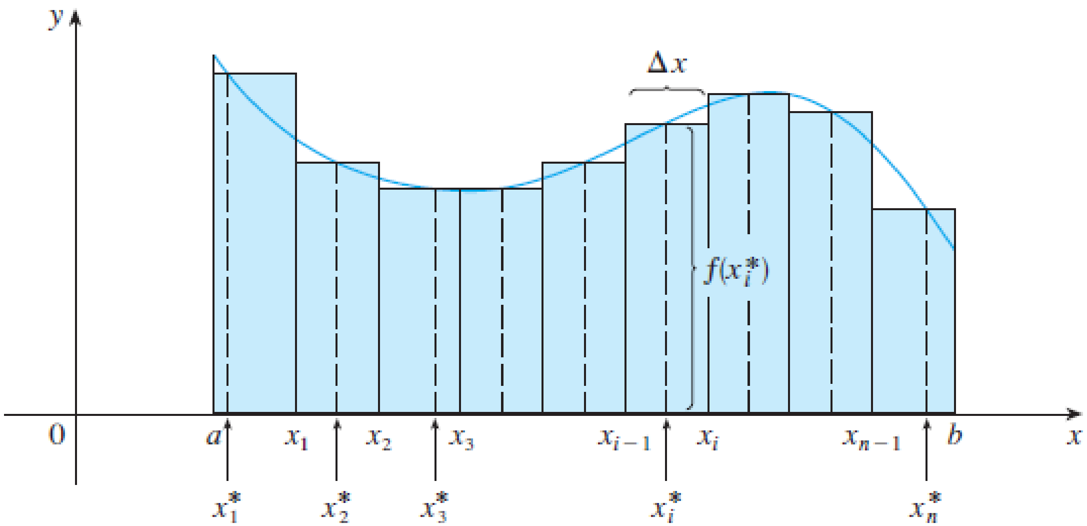


FIGURE 1

In a similar manner we consider a function of two variables defined on a closed rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, c \le y \le d\}$$

and we first suppose that $f(x,y) \ge 0$. The graph of f is a surface with equation $z = f(x,y)$.

Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x,y,z) \in \mathbb{R}^3 : 0 \le z \le f(x,y), (x,y) \in \mathbb{R}\}$$

(See Figure 2.) Our goal is to find the volume of S.

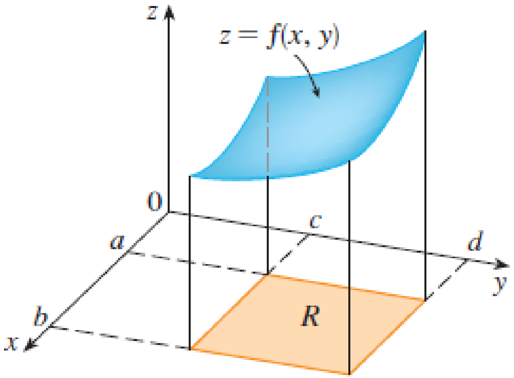
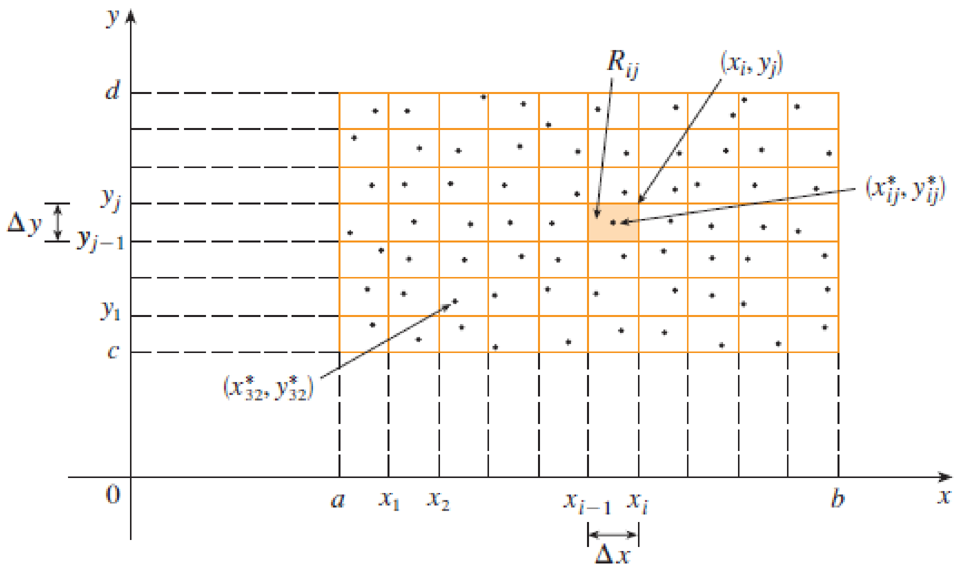


FIGURE 2

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{m}$ and dividing $[c, d]$ into n subintervals $[y_{i-1}, y_i]$ of equal width $\Delta y = \frac{d-c}{n}$.

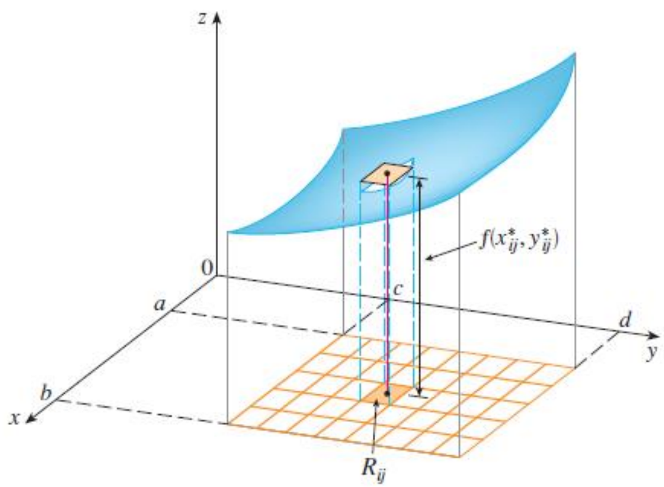
By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in the following figure.



We form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y): x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$$

each with area $\Delta A = \Delta x \Delta y$.

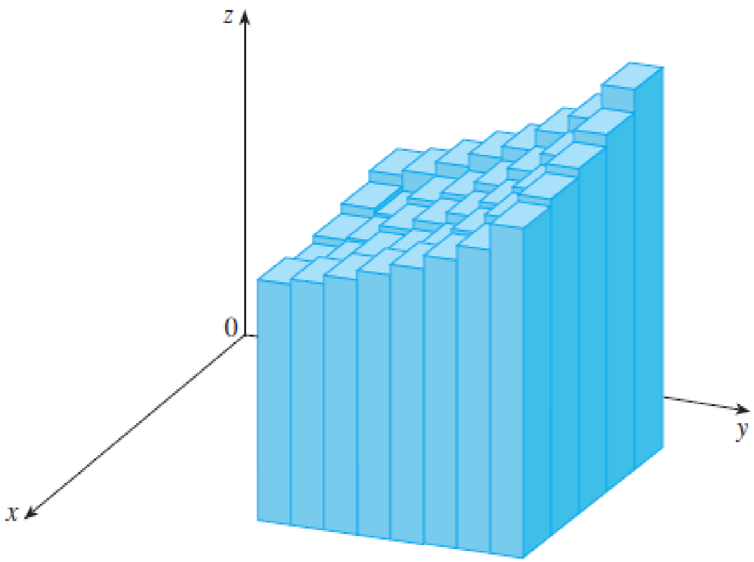


If we choose a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij} , then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in the Figure 4. The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$



This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results. Our intuition tells us that the approximation given in becomes better as m and n become larger and so we would expect that

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

We use the expression in this equation to define the volume of the solid S that lies under the graph of f and above the rectangle R. So we give the following definition:

Definition:

The double integral of f over the rectangle R is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

İf this limit exists.

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

4 **Fubini's Theorem** If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Fubini's Theorem –Stronger form

Let $f(x,y)$ be continuous on a region R

- If R is defined by $a \leq x \leq b,\ u(x) \leq y \leq v(x)$, with u and v are continuous on $[a,b]$, then

$$\iint\limits_R f(x,y)dA = \int_a^b \int_{u(x)}^{v(x)} f(x,y)dydx.$$

- If R is defined by $c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)$ with h_1 and h_2 are continuous on $[c,d]$, then

$$\iint\limits_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

Properties of Double Integrals

$$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$$\iint_D c\,f(x,y)\,dA = c\iint_D f(x,y)\,dA$$

If $D = D_1 \cup D_2$, $\displaystyle\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$

If $f(x,y) \geqslant g(x,y)$ for all (x,y) in D, then $\displaystyle\iint_D f(x,y)\,dA \geqslant \iint_D g(x,y)\,dA$