Physics 122: Electricity & Magnetism – Lecture 5 Gauss’s Law

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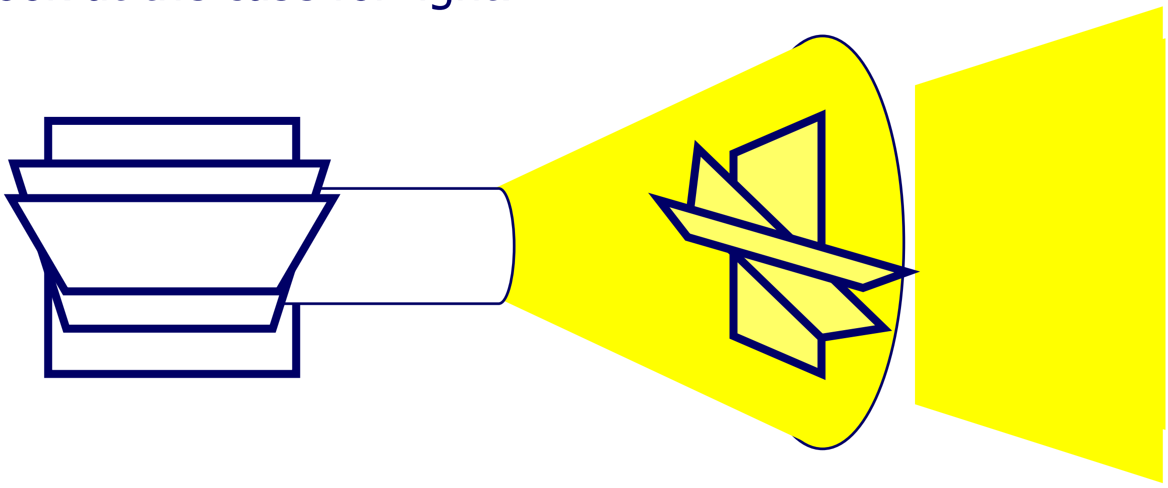
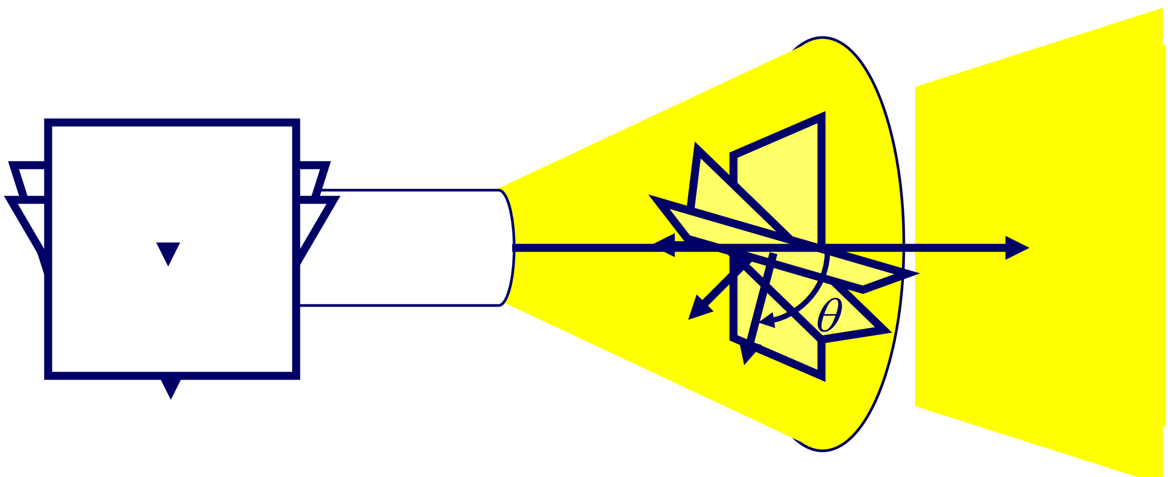
Flux

- Flux in Physics is used to two distinct ways.
- The first meaning is the rate of flow, such as the amount of water flowing in a river, i.e. volume per unit area per unit time. Or, for light, it is the amount of energy per unit area per unit time.
- Let's look at the case for light:



Area Vector

- Represent an area as a vector $\Delta\vec{A}$, of length equal to the area, and direction of the "outward normal" to the surface.
- The flux of light through a hole of area ΔA is proportional to the area, and the cosine of the angle between the light direction and this area vector.



- If we use a vector $\vec{L}$ to represent the light energy per unit time, then the light out of the hole is $L\Delta A\cos\theta = \vec{L}\cdot\Delta\vec{A}$. In this case it is *negative* $(\theta > 90°)$ which means the light flux is into the hole.

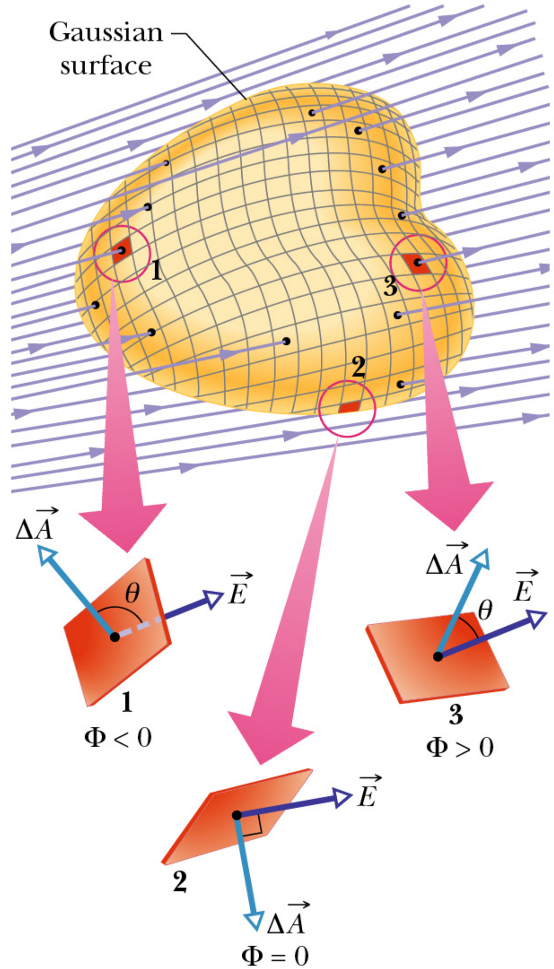
Flux of Electric Field

- Like the flow of water, or light energy, we can think of the electric field as flowing through a surface (although in this case nothing is actually moving).
- We represent the flux of electric field as Φ (greek letter phi), so the flux of the electric field through an element of area ΔA is

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{A} = E\,\Delta A\cos\theta$$

- When $\theta < 90^\circ$, the flux is positive (out of the surface), and when $\theta > 90^\circ$, the flux is negative.
- When we have a complicated surface, we can divide it up into tiny elemental areas:

$$d\Phi = \vec{E}\cdot d\vec{A} = E\,dA\cos\theta$$



Gauss' Law

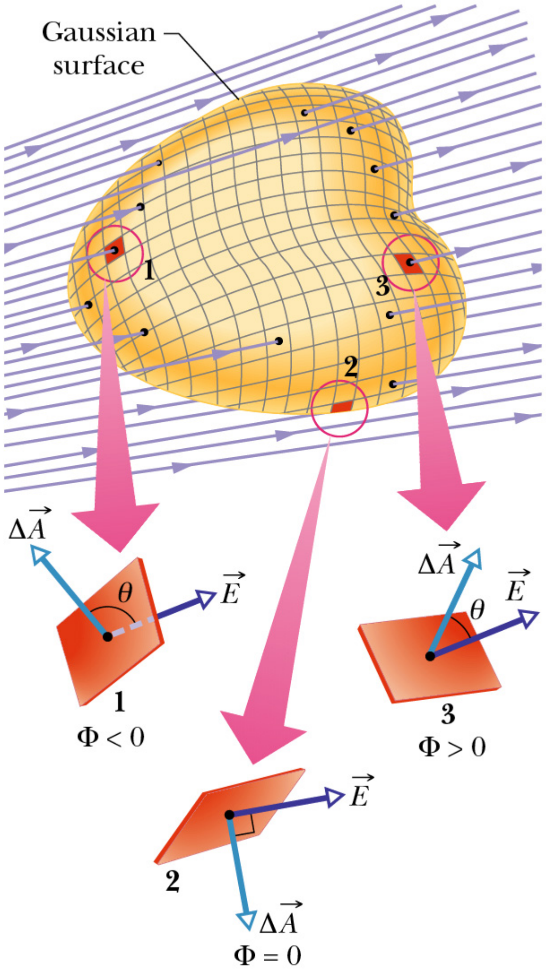
- We are going to be most interested in *closed* surfaces, in which case the outward direction becomes self-evident.
- We can ask, what is the electric flux out of such a closed surface? Just integrate over the closed surface: $\Phi = \oint d\Phi = \oint \vec{E} \cdot d\vec{A}$

Flux positive => out
Flux negative => in

- The $\oint$ symbol has a little circle to indicate that the integral is over a closed surface.
- The closed surface is called a gaussian surface, because such surfaces are used by Gauss' Law, which states that:

Gauss' Law

The flux of electric field through a closed surface is proportional to the charge enclosed.



Mathematical Statement of Gauss' Law

- The constant of proportionality in Gauss' Law is our old friend ε_0.

$$\varepsilon_0 \Phi = q_{enc}$$

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc}$$

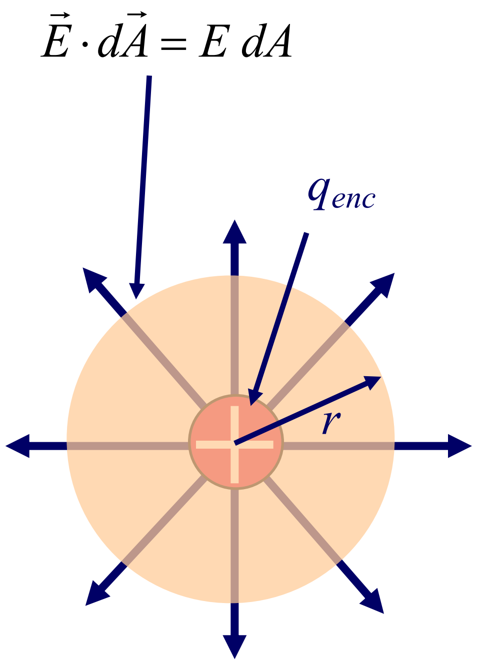
- Recall that I said that we would see later why Coulomb's constant is written $k_E = \dfrac{1}{4\pi\varepsilon_0}$?

- We can see it now by integrating the electric flux of a point charge over a spherical gaussian surface.

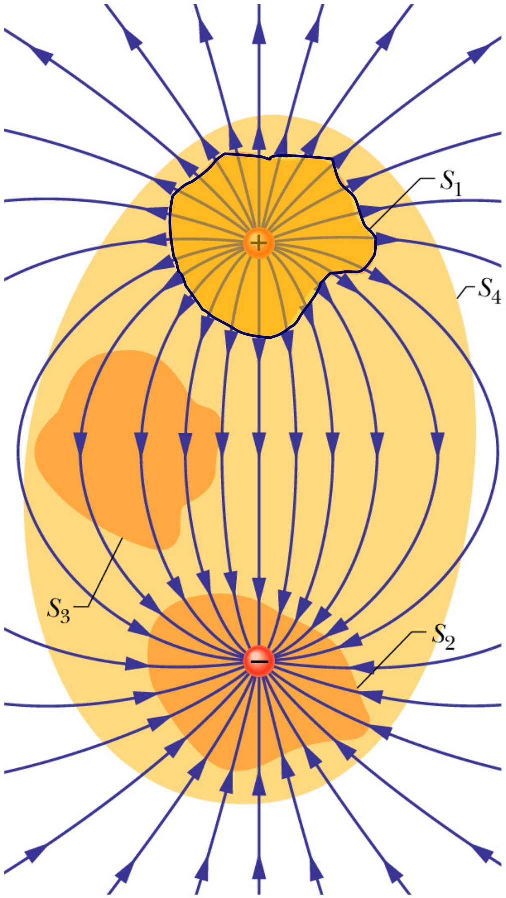
$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E \oint dA = \varepsilon_0 E 4\pi r^2 = q_{enc}$$

- Solving for E gives Coulomb's Law.

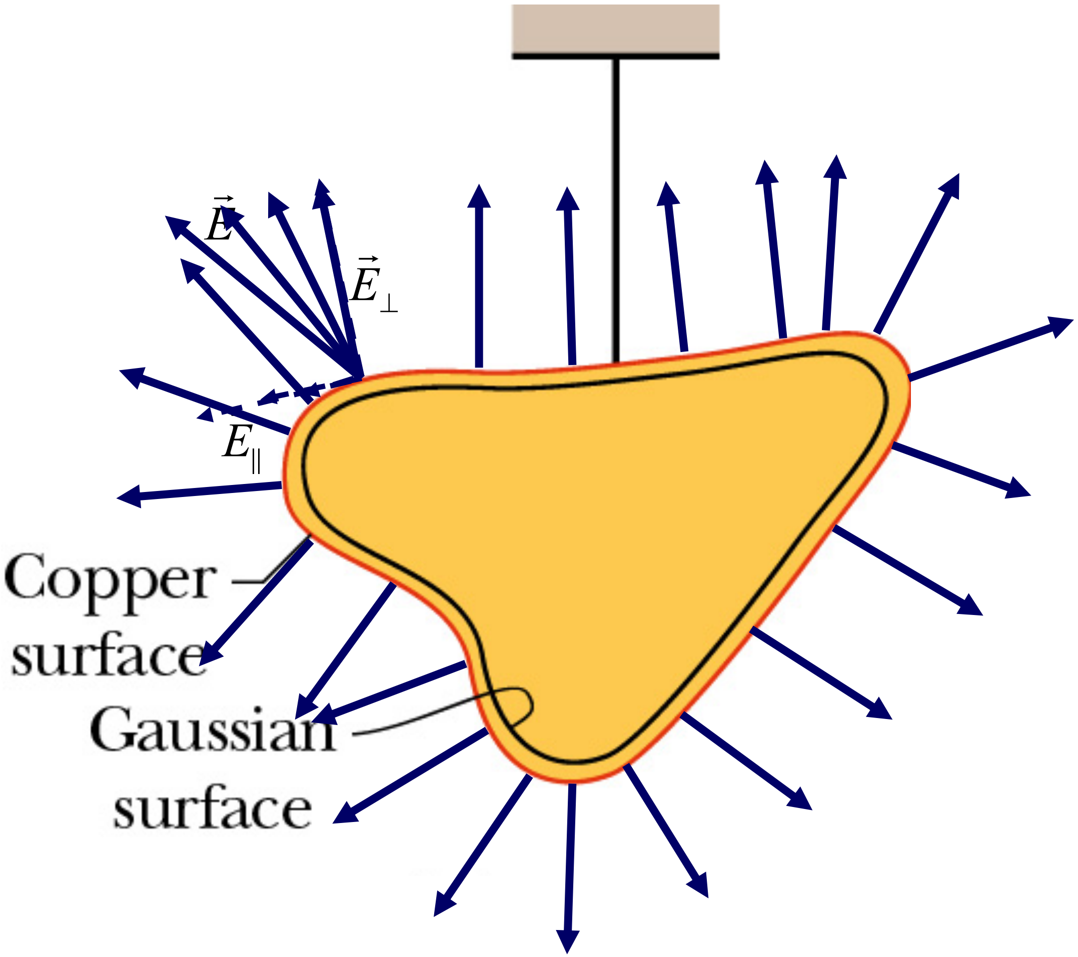
$$E = \frac{1}{4\pi\varepsilon_0}\frac{q_{enc}}{r^2}$$



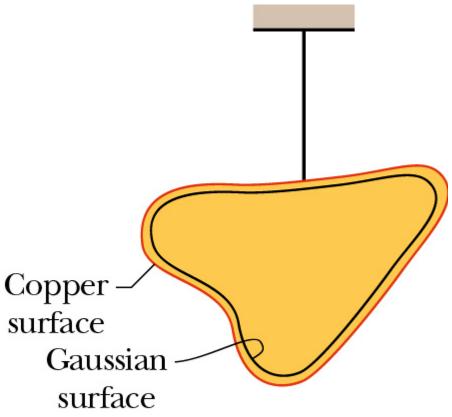
Example of Gauss’ Law



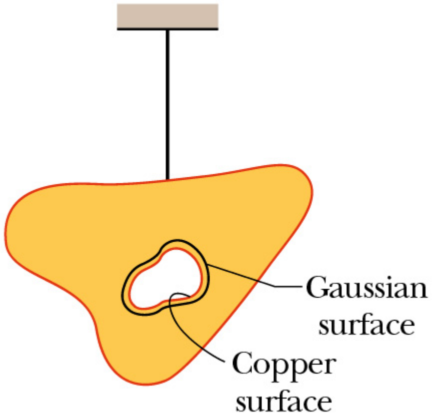
Field At the Surface of a Conductor



Field Inside a Conductor



(*a*)



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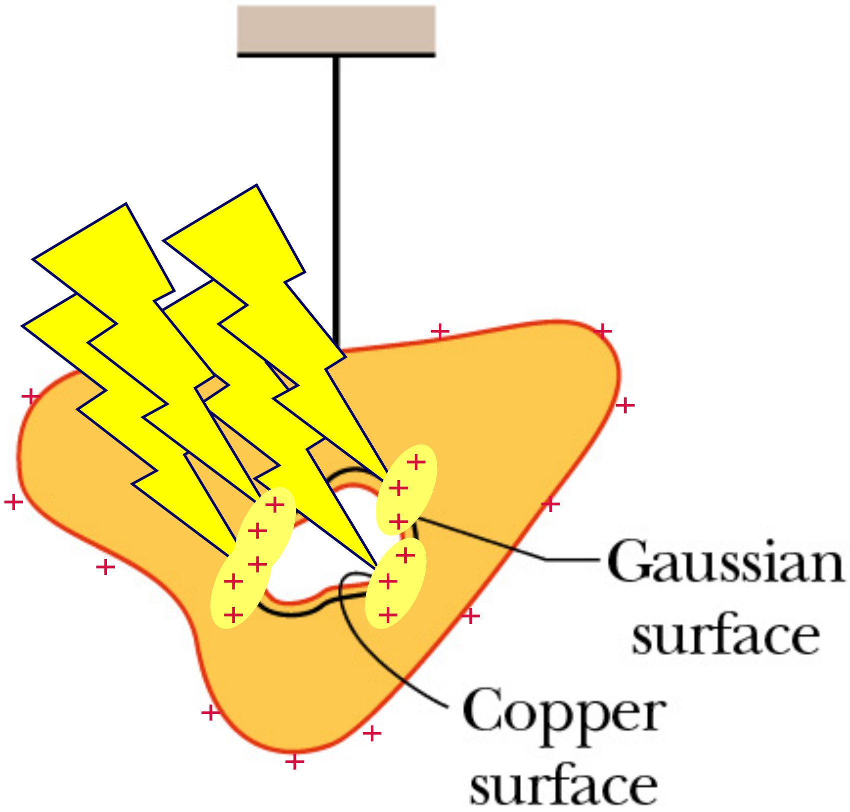
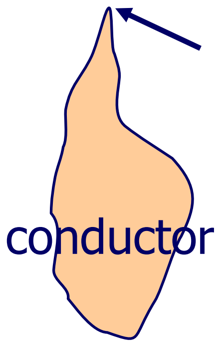
Field Inside a Conductor



- We have the remarkable fact that if you try to deposit charge on the inside of the conductor...
- The charges all move to the outside and distribute themselves so that the electric field is everywhere normal to the surface.
- This is NOT obvious, but Gauss' Law allows us to show this!

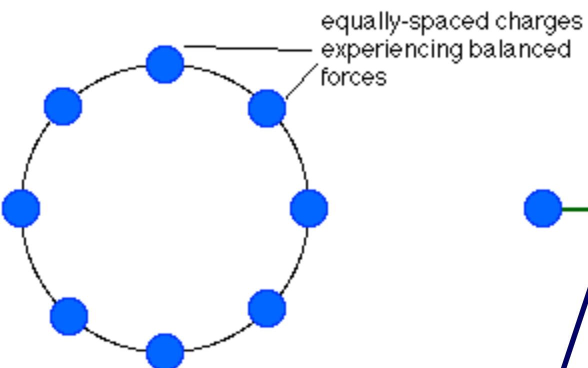
There are two ideas here
- Electric field is zero inside conductors
- Because that is true, from Gauss' Law, cavities in conductors have E = 0

Charge Distribution on Conductors

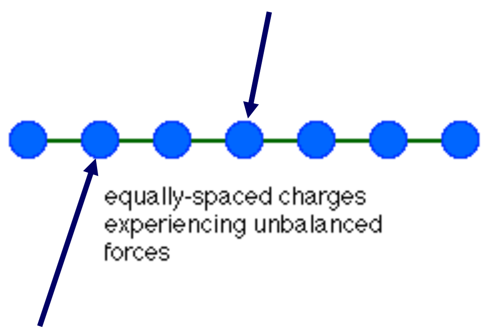


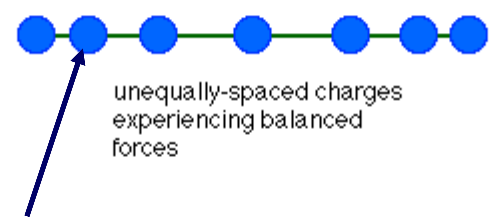
charges collect near tip

- For a conducting sphere, the charges spread themselves evenly around the surface.
- For other shapes, however, the charges tend to collect near sharp curvature.
- To see why, consider a line of charge.



balanced forces

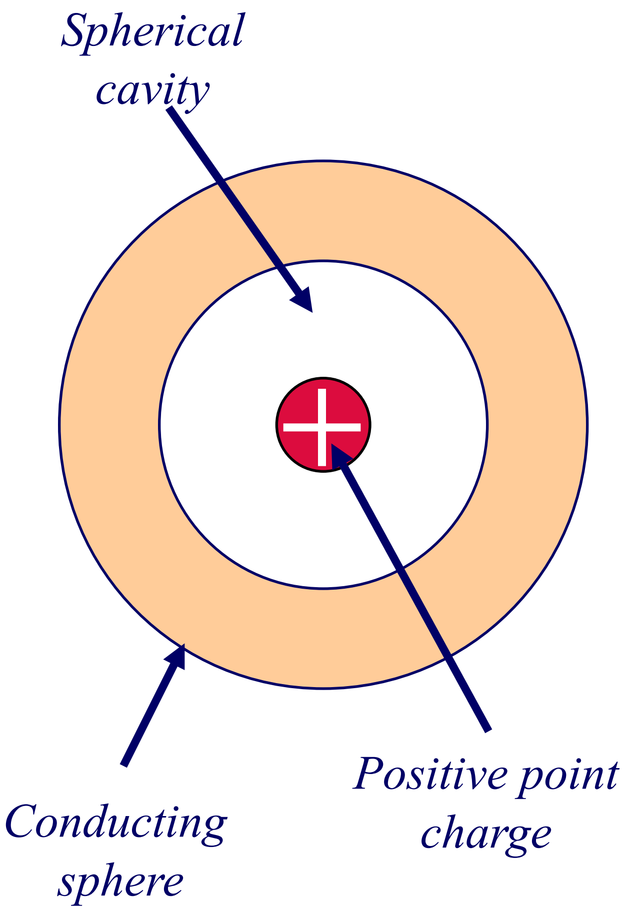




unbalanced forces (pushed on by one charge from left, but by 5 charges from right)

redistributed charges (pushed on by one nearby charge from left, but by 5 more distant charges from right)

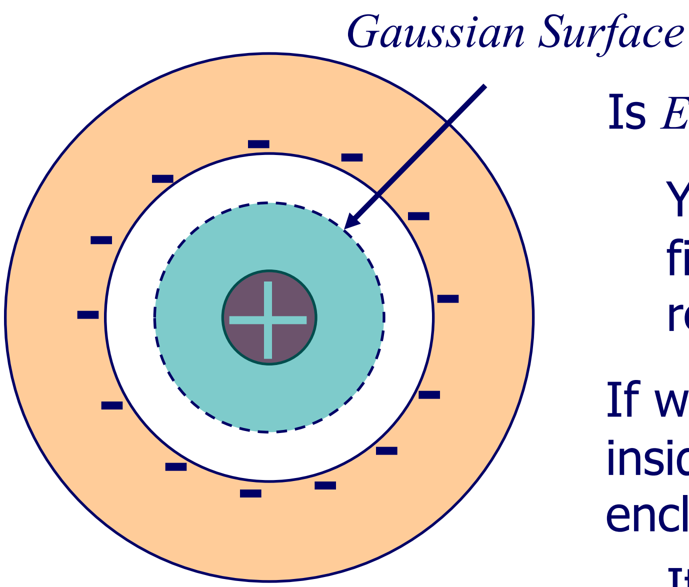
A Charge Inside a Conductor



2. What will happen when we add a charge inside a conductor?
   A. E field is still zero in the cavity.
   B. E field is not zero in the cavity, but it is zero in the conductor.
   C. E field is zero outside the conducting sphere.
   D. E field is the same as if the conductor were not there (i.e. radial outward everywhere).
   E. E field is zero in the conductor, and negative (radially inward) outside the conducting sphere.

Use Gauss' Law to Find Out

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc}$$



Is $E = 0$ in the cavity?

No, because there is charge enclosed (Gauss' Law).

Is $E = 0$ in the conductor?

Yes, because as before, if there were an electric field in the conductor, the charges would move in response (NOT Gauss' Law).

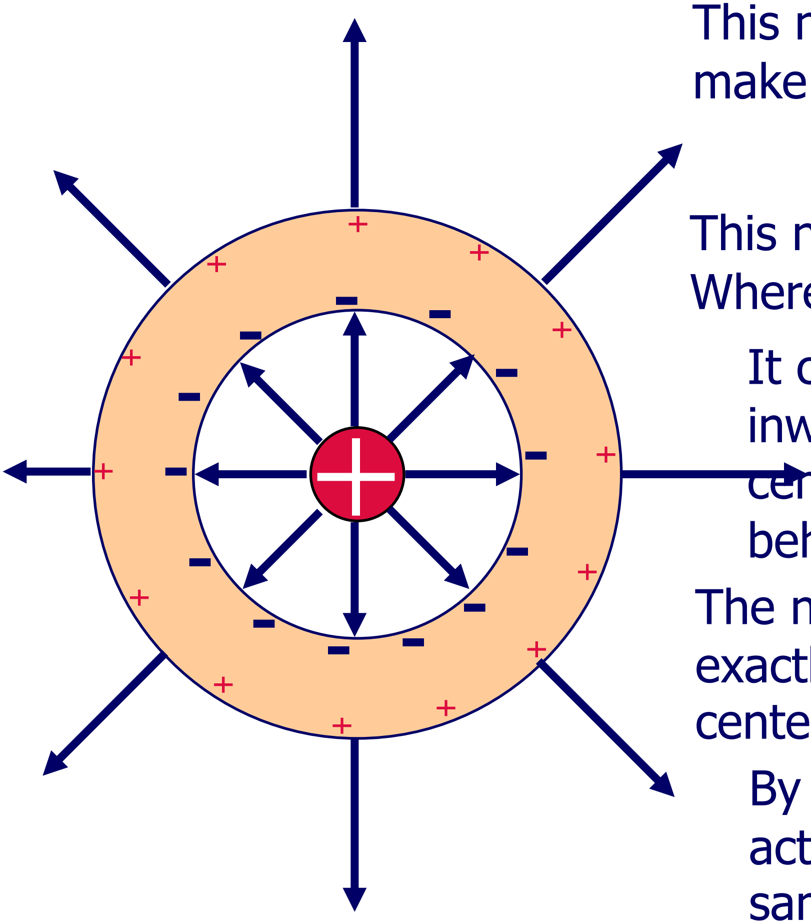
If we enlarge the gaussian surface so that it is inside the conductor, is there any net charge enclosed?

It looks like there is, but there cannot be, because Gauss' Law says $E = 0$ implies $q_{\text{enc}} = 0$!

How do we explain this?

There must be an equal and opposite charge induced on the inner surface.

E Field of Charge In Conductor



This negative charge acts with the inner charge to make the field radial inside the cavity.

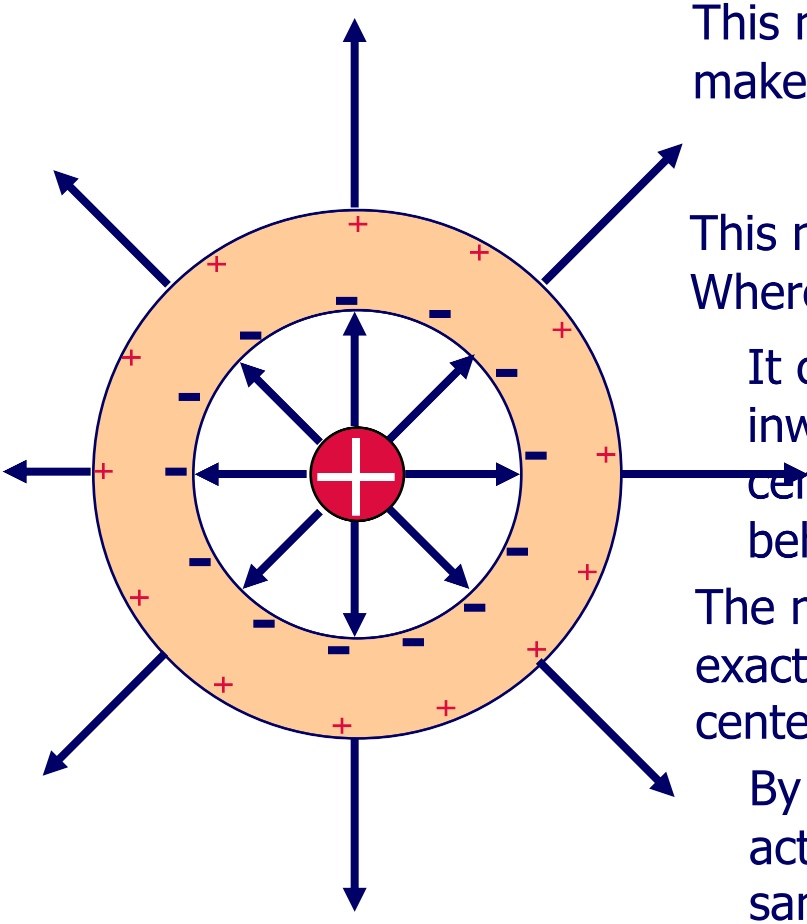
This negative charge cannot appear out of nowhere. Where does it come from?

It comes from the outer surface (electrons drawn inward, attracted to the positive charge in the center). Therefore, it leaves positive charge behind.

The net positive charge that appears conductor is exactly the same as the original charge in the center, so what do the field lines look like?

By spherical symmetry, the positive shell of charge acts like a point charge in the center, so field is the same as the field of the original point charge.

E Field of Charge In Conductor



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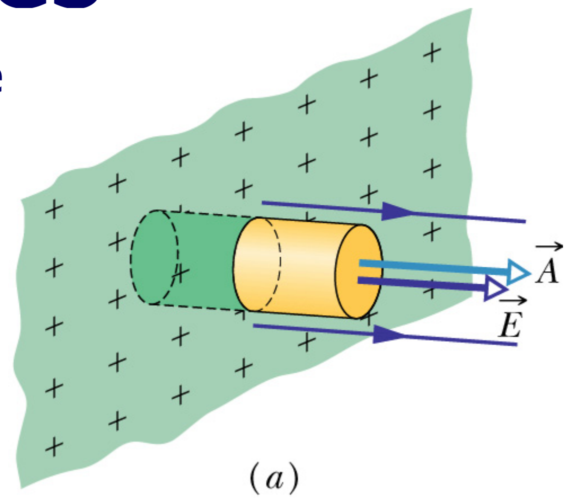
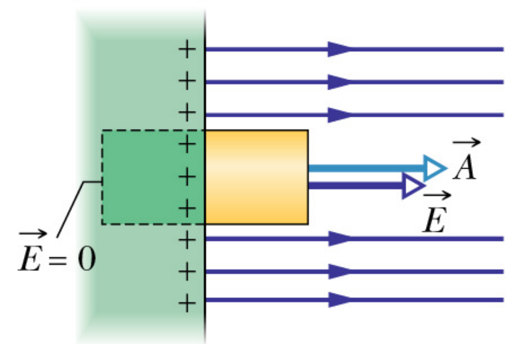
Other Geometries

- Always use the symmetry of the problem to determine what shape to make your gaussian surface.
- Here is a plate (plane) geometry, where the charges are evenly distributed on a flat surface. If the total charge on the plate is Q, and the plate has a total area A_{tot}, then the *surface charge density* is

$$\sigma = Q / A_{\text{tot}} \quad \text{C/m}^2$$



- The E field is everywhere perpendicular to the plate (again, if not, the charges will move until the part parallel to the surface is nullified). What is $\vec{E}$?
- Use a gaussian surface that is parallel to $\vec{E}$ on the sides (so no flux through side surfaces), and closes inside the conductor (no flux through that end).
- On the remaining side, the area vector $\vec{A}$ is parallel to the E field, so $\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E A = \sigma A$ or $E = \dfrac{\sigma}{\varepsilon_0}$



Conducting Surface