EEE328
Digital Signal Processing

Ankara University

Faculty of Engineering

Electrical and Electronics Engineering Department

Discrete-Time Fourier Transform

EEE328 Digital Signal Processing

Lecture 14

Agenda

- Discrete-Time Fourier Transform
- Discrete Fourier Series
- Properties of Discrete Fourier Series
- Fourier Transform of Periodic Signals
- Sampling The Fourier Transform
- Fourier Representation of Finite-Duration Sequences: The Discrete Fourier Transform
- Discrete Cosine Transform
- Efficient Computation of The Discrete Fourier Transform
- Fourier Analysis of Signals Using The DFT

Discrete Fourier Series

$$e_k[n] = e^{j\left(\frac{2\pi}{N}\right)kn} = e_k[n + rN]$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

$$e_{k+lN}[n] = e^{j\left(\frac{2\pi}{N}\right)(k+lN)n} = e^{j\left(\frac{2\pi}{N}\right)kn} e^{j2\pi ln} = e^{j\left(\frac{2\pi}{N}\right)kn} = e_k[n]$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

Synthesis Equation

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, W_N^{-kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

Analysis Equation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, W_N^{kn}$$

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

Properties of Discrete Fourier Series

Linearity

$$\tilde{x}_1[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]$$

$$\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_2[k]$$

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

Properties of Discrete Fourier Series

Shifting

$$\tilde{x}[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\tilde{X}[k]$$

$$W_N^{-nl}\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k-l]$$

Duality

$$N\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k]W_N^{kn}$$

$$N\tilde{x}[-k] = \sum_{n=0}^{N-1} \tilde{X}[n]W_N^{nk}$$

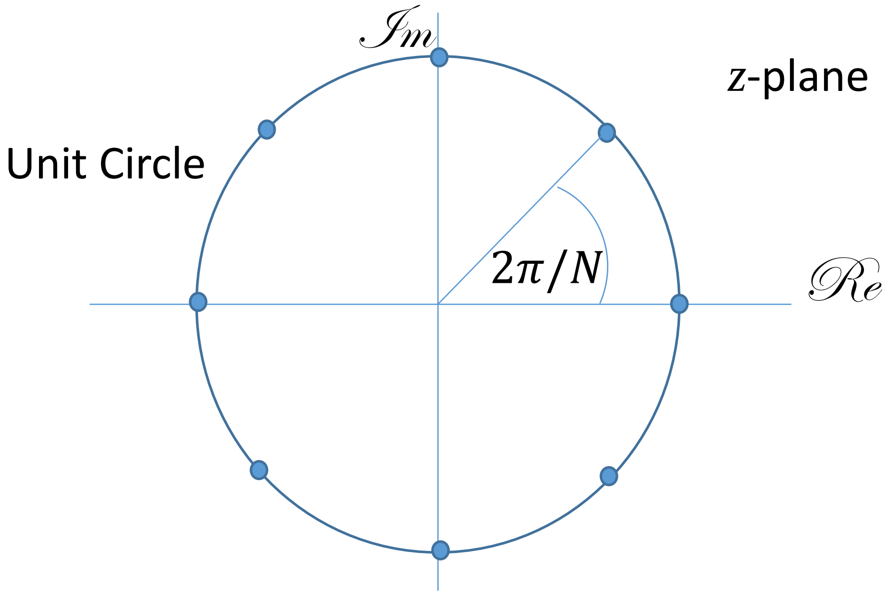
$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k]$$

$$\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]$$

Fourier Transform of Periodic Signals

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k] \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Sampling The Fourier Transform



Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ (N=8).

Fourier Representation of Finite-Duration Sequences: The Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

Analysis Equation

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] W_N^{-kn}$$

Synthesis Equation

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$$

Discrete Cosine Transform (DCT)

DCT Type-I

$$X^{c1}[k] = 2\sum_{n=0}^{N-1} \alpha[n]x[n]\cos\left(\frac{\pi k n}{N-1}\right), \qquad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N-1}\sum_{k=0}^{N-1} \alpha[k]X^{c1}[k]\cos\left(\frac{\pi k n}{N-1}\right), \qquad 0 \le n \le N-1$$

$$x[n] \overset{\mathcal{DCT}-1}{\longleftrightarrow} X^{c1}[k]$$

$$\alpha[n] = \begin{cases} \frac{1}{2}, & n = 0 \; and \; N-1 \\ 1, & 1 \le n \le N-2 \end{cases}$$

Discrete Cosine Transform (DCT)

DCT Type-II

$$X^{c2}[k] = 2\sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right), \qquad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \beta[k]X^{c2}[k]\cos\left(\frac{\pi k(2n+1)}{2N}\right), \qquad 0 \le n \le N-1$$

$$x[n] \overset{\mathcal{DCT}-2}{\longleftrightarrow} X^{c2}[k]$$

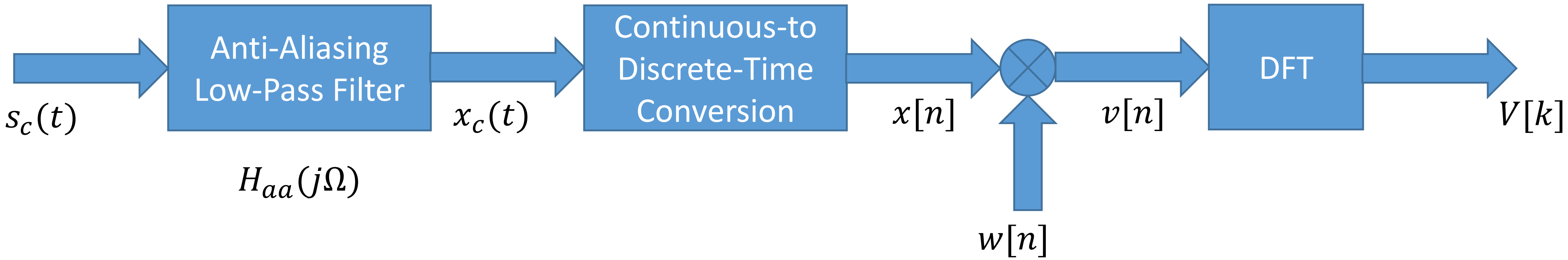
$$\beta[k] = \begin{cases} \frac{1}{2}, & k = 0 \\ 1, & 1 \le k \le N-1 \end{cases}$$

Efficient Computation of The Discrete Fourier Transform

Most approached to improve the efficiency of the computation of the DFT exploit the symmetry and periodicity properties of W_N^{kn}, specifically,

1. $W_N^{k[N-n]} = W_N^{-kn} = (W_N^{kn})^*$ (complex conjugate symmetry),
2. $W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$ (periodicity in *n* and *k*).

Fourier Analysis of Signals Using The DFT



Processing steps in the Discrete-Time Fourier analysis of a continuous-time signal

References

- Signals & Systems, Second Edition, A. V. Oppenheim, A. S. Willsky with S. H. Nawab, Prentice Hall, 1997
- Discrete-Time Signal Processing, Second Edition, A. V. Oppenheim, R. W. Schafer with J. R. Buck, Prentice Hall, 1999