Vector Operations

Scalars: magnitude but no direction;
mass, charge, density, temperature, etc.

Vectors: magnitude and direction,
displacement, velocity, acceleration, force, momentum, etc.

Addition of two vectors;

Figure 1 - 3

commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

multiplication by scalar is *distributive:* $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$

Dot product of two vectors: is itself a scalar (hence the alternative name ***scalar product***).

$$\mathbf{A} \cdot \mathbf{B} \equiv AB\cos\theta$$

Figure 6

The dot product is *commutative;* $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

It is *distributive;* $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$

$\mathbf{A} \cdot \mathbf{A} = A^2$ If **A** and **B** are perpendicular, then $\mathbf{A} \cdot \mathbf{B} = 0$.

E.g.: Let $\mathbf{C} = \mathbf{A} - \mathbf{B}$, calculate the dot product of **C** with itself.

$$\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B}$$

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

This is the **law of cosines**.

Cross product of two vectors:

$$\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta \, \hat{\mathbf{n}}$$

n is a **unit vector** (vector of magnitude 1) pointing perpendicular to the plane of **A** and **B**.

A × **B** is itself a *vector* (hence the alternative name **vector product**).

The cross product is *distributive;* $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$

It is *not commutative;* $(\mathbf{A} \times \mathbf{B}) \neq (\mathbf{B} \times \mathbf{A})$ $\quad$ $(\mathbf{A} \times \mathbf{B}) = -(\mathbf{B} \times \mathbf{A})$

Vector Algebra: *Component Form*

Cartesian coordinates *x*, *y*, *z*

$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$; unit vectors parallel to the *x*, *y*, and *z* axes

A; an arbitrary vector $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$

Figure 9

A_x , A_y , and A_z , are the "components" of **A**;

$$\mathbf{A} + \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) + (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

To multiply by a scalar, multiply each component; $a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}$

$$\underline{\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1} \qquad \underline{\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0}$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1 \qquad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0$$

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) \cdot (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned}$$

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2 \qquad A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Vector product

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = \mathbf{0}$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} = \hat{\mathbf{z}}$$

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$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \hat{\mathbf{y}}$$

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}\end{aligned}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Scalar triple product: A · (B × C)

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

"alphabetical" order is preserved!

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector triple product: A × (B × C).

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \Longrightarrow$$ **BAC-CAB** rule.

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=-\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=-\mathbf{A}(\mathbf{B}\cdot\mathbf{C})+\mathbf{B}(\mathbf{A}\cdot\mathbf{C})$$

Position, Displacement, and Separation Vectors

The vector from the origin (O) to a point is called **position vector**

$$\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

Figure 13

Its magnitude, the distance from the origin;

$$r = \sqrt{x^2 + y^2 + z^2}$$

Its unit vector, (pointing radially outward);

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$;

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

(or is called $d\mathbf{r}$)

source point, **r**, where an electric charge is located, and a
field point, **r**, at which you are calculating the electric or magnetic field.

Separation vector from the source point to the field point. I shall use for this purpose the script letter $\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'$$

Its magnitude is

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$$

Figure 14

The unit vector in the direction from $\mathbf{r}'$ to $\mathbf{r}$ is

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

“Ordinary” Derivatives

Suppose we have a function of one variable: *f (x)*.

What does the derivative, *df/dx*, do for us?

Answer: It tells us how rapidly the function *f (x)* varies when we change the argument *x* by a tiny amount, *dx*:

$$df = \left(\frac{df}{dx}\right) dx$$

Geometrical Interpretation: The derivative *d f/dx* is the *slope* of the graph of *f* versus *x*.

Figure 17

Gradient

How fast the temperature *T (x, y, z)* in this room does vary?

A theorem on partial derivatives states that

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

It is reminiscent (reminding) of a dot product:

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\boldsymbol{\nabla} T) \cdot (d\mathbf{l}) \end{aligned}$$

where

$$\boldsymbol{\nabla} T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

is the **gradient** of *T* .

The Del Operator

The gradient has the formal appearance of a vector, ∇, "multiplying" a scalar T :

$$\nabla T = \underbrace{\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right)} T$$

It is called **del**. $\longrightarrow$ $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$

del is *not* a vector, in the usual sense. ∇ is a **vector operator** that *acts upon* T, not a vector that multiplies T .

There are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);

2. On a vector function **v**, via the dot product: $\nabla \cdot \mathbf{v}$ (the **divergence**);

3. On a vector function **v**, via the cross product: $\nabla \times \mathbf{v}$ (the **curl**).

The Divergence

From the definition of ∇ we construct the divergence:

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.\end{aligned}$$

Geometrical Interpretation: The name **divergence** is well chosen, for $\nabla \cdot \mathbf{v}$ is a measure of how much the vector **v** spreads out (diverges) from the point in question.

Figure 18

The Curl

From the definition of **∇,** the curl:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Geometrical Interpretation: The **curl;** for $\nabla \times \mathbf{v}$ is a measure of how much the vector **v** swirls around the point in question.

Figure 19