The Curl

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

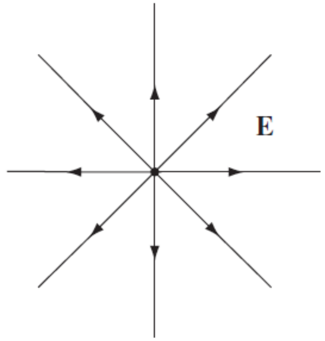
Geometrical Interpretation: $\nabla \times \mathbf{v}$ is a measure of how much the vector **v** swirls around the point in question.

Figure

The Curl of E

The ***E*** field of a point charge at the origin:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$



the curl of this field is zero

$$\nabla \times \mathbf{E} = \mathbf{0}$$

The line integral of a field from a point a to point b:

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

Figure 29

r_a ; the distance from the origin to the point a

r_b ; the distance to b.

In spherical coordinates;

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} \quad \Longrightarrow \quad \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

($\mathbf{E}$ is in radial direction, θ ve φ don't contribute to **E . d*l***)

Therefore;

$$\int_a^b \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}dr = \left.\frac{-1}{4\pi\epsilon_0}\frac{q}{r}\right|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

Result: The line integral only depends on the coordinates of the endpoints; that is, independent of the path.

The integral around a closed path $r_a = r_b$:

Figure 29

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0$$

Stokes' theorem:

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Applying Stokes' theorem;

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \longrightarrow \quad \nabla \times \mathbf{E} = \mathbf{0}$$

(hold for *any static charge distribution whatever.)*

For many charges, using the principle of superposition;

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots$$

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \ldots = \mathbf{0}$$

ELECTRIC POTENTIAL

Any vector whose curl is zero is equal to the gradient of some scalar.

$$\nabla \times \mathbf{F} = \mathbf{0} \Longleftrightarrow \mathbf{F} = -\nabla V$$

Theorem:
Curl-less (or "**irrotational**") **fields**. The following conditions are equivalent (that is, **F** satisfies one if and only if it satisfies all the others):

(a) $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere.

(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of path, for any given end points.

(c) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.

(d) **F** is the gradient of some scalar function: $\mathbf{F} = -\nabla V$.

Because the line integral is independent of path, we can define a function:

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

$\mathcal{O}$: standard reference point. It is called the **electric potential**.

The potential *difference* between two points **a** and **b** is

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

The fundamental theorem for gradients states that

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

$$\mathbf{E} = -\nabla V$$

Example 7. Find the potential inside and outside a spherical shell of radius R that carries a uniform surface charge. Set the reference point at infinity.

Using Gauss's law, the field outside is

Figure 31

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

The field inside the shell is zero.

For points outside the sphere ($r > R$),

$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

The potential inside the sphere ($r < R$),

$$V(r) = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{R}\frac{q}{r'^2}\,dr' - \int_{R}^{r}(0)\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{R} + 0 = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

The potential is *not* zero inside the shell, even though the field is.
V is a *constant* in this region, to be sure, so that $\nabla V = \mathbf{0}$—that's what matters.

Poisson's Equation and Laplace's Equation

The electric field written as the gradient of a scalar potential;

$$\mathbf{E} = -\nabla V$$

What do the divergence and curl of **E**, look like, in terms of *V*?

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = \mathbf{0}$$

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$ → **Poisson's equation**

In regions where there is no charge, so $\rho = 0$,

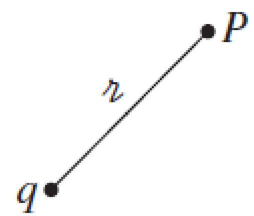
$$\nabla^2 V = 0$$ → **Laplace's equation**

The curl of **E**;

$$\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = \mathbf{0}$$

The Potential of a Localized Charge Distribution

The electric field of a point charge at the origin:



$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

In spherical coordinates;

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} \quad \Longrightarrow \quad \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

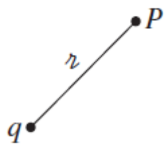
Setting the reference point at infinity, the potential of a point charge q at the origin is;

$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

???

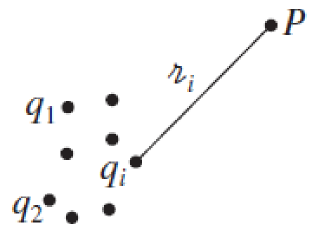
Figure 32

point charge



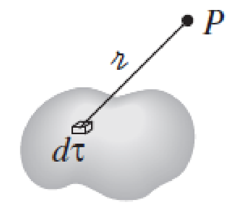
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

collection of charges;



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$

continuous distribution



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}dq$$

For a volume charge

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

The potentials of line and surface charges:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,dl'$$

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,da'$$

Example 8. Find the potential of a uniformly charged spherical shell of radius *R*.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}}\, da'$$

Figure 33

using the law of cosines to:

$$da' = R^2 \sin\theta'\, d\theta'\, d\phi'$$

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$$

$$V(z) = \frac{\sigma}{4\pi\epsilon_0}\int \frac{R^2 \sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$\begin{aligned}
4\pi\epsilon_0 V(z) &= \sigma \int \frac{R^2 \sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} \\
&= 2\pi R^2 \sigma \int_0^{\pi} \frac{\sin\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}\, d\theta' \\
&= 2\pi R^2 \sigma \left(\frac{1}{Rz}\sqrt{R^2 + z^2 - 2Rz\cos\theta'} \right)\Bigg|_0^{\pi} \\
&= \frac{2\pi R\sigma}{z}\left(\sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz} \right) \\
&= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2} \right].
\end{aligned}$$

$$4\pi\epsilon_0 V(z) = \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right].$$

For points *outside* the sphere, *z* is greater than *R* → $\sqrt{(R-z)^2} = z - R$

For points *inside* the sphere → $\sqrt{(R-z)^2} = R - z$

outside the sphere; $V(z) = \frac{R\sigma}{2\epsilon_0 z}[(R+z) - (z-R)] = \frac{R^2\sigma}{\epsilon_0 z}$

inside the sphere; $V(z) = \frac{R\sigma}{2\epsilon_0 z}[(R+z) - (R-z)] = \frac{R\sigma}{\epsilon_0}$

In terms of *r* and the total charge on the shell, $q = 4\pi R^2\sigma$,

$$V(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r} & (r \geq R), \\ \frac{1}{4\pi\epsilon_0}\frac{q}{R} & (r \leq R). \end{cases}$$

The three fundamental quantities of electrostatics: ρ, $\mathbf{E}$, and V :

Figure 35