Boundary Conditions

Gauss's law;

Figure 36

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A$$

($E^{\perp}_{\text{above}}$ denotes the component of **E** that is perpendicular to the surface immediately above)

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0} \sigma$$

The normal component of **E** *is discontinuous by an amount* σ/ϵ_0 *at any boundary.*

The *tangential* component of **E**, is *always* continuous.

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

Figure 37

as $\epsilon \to 0$ ⟹ $E^{\parallel}_{\text{above}} l - E^{\parallel}_{\text{below}} l$

$$\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}$$

$\mathbf{E}^{\parallel}$ stands for the components of **E** *parallel* to the surface.

The boundary conditions on **E** can be combined into a single formula:

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

$\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface.

The potential, meanwhile, is continuous across *any* boundary, since

$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

Figure 38

as the path length shrinks to zero, the integral:

$$V_{\text{above}} = V_{\text{below}}$$

However, the *gradient* of *V* inherits the discontinuity in **E**; since $\mathbf{E} = -\nabla V$,

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0}\sigma\,\hat{\mathbf{n}}$$

or,

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0}\sigma \quad \Longrightarrow \quad \text{where} \quad \frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

denotes the **normal derivative** of *V* (that is, the rate of change in the direction perpendicular to the surface).

WORK AND ENERGY IN ELECTROSTATICS

The Work It Takes to Move a Charge:

At stationary of source charges, moving a test charge Q from point **a** to point **b** : How much work will you have to do?

Figure 39

At any point along the path, the electric force on Q is **F** = Q**E**;
the force you must exert, in opposition to this electrical force, is −Q**E**.

The work you do is therefore

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}$$

(*the potential difference between points* **a** *and* **b** *is equal to the work per unit charge required to carry a particle from* **a** *to* **b**.)

if you bring Q in from far away and stick it at point $\mathbf{r}$, the work you must do is

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

if you have set the reference point at infinity;

$$W = QV(\mathbf{r}) \Longrightarrow V(\mathbf{r}) = \frac{W}{Q}$$

Potential is potential *energy* (the work it takes to create the system) *per unit charge* (just as the *field* is the *force* per unit charge).

The Energy of a Point Charge Distribution

How much work would it take to assemble an entire *collection* of point charges?

Figure 40

The first charge, q_1, takes *no* work, since there is no field yet to fight against.
Now bring in q_2.

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}} \right)$$

Figure 36

Now, bring in q_3;

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}} \right)$$

the extra work to bring in q_4

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}} \right)$$

The *total* work necessary to assemble the first four charges;

$$W = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}} \right)$$

the general rule:

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

count each pair twice, and then divide by 2:

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)}$$

the potential at point $\mathbf{r}_i$ (the position of q_i) due to all the *other* charges

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$$

The Energy of a Continuous Charge Distribution

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$$

For a volume charge density ρ, → $W = \frac{1}{2}\int \rho V \, d\tau$

(For line and surface charges:
$\int \lambda V \, dl$ $\quad$ $\int \sigma V \, da$)

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

$$W = \frac{\epsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V \, d\tau$$

using $\nabla . (\boldsymbol{E}V) = (\nabla . \boldsymbol{E})\, V + \mathbf{E} . (\nabla V)$

Divergence theorem

$$W = \frac{\epsilon_0}{2}\left[-\int \mathbf{E} \cdot (\nabla V)\, d\tau + \oint V\mathbf{E} \cdot d\mathbf{a}\right]$$

$\nabla V = -\mathbf{E}$ → $W = \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V\mathbf{E} \cdot d\mathbf{a}\right)$

$$W = \frac{1}{2}\int \rho V \, d\tau \quad \longrightarrow \quad W = \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V\mathbf{E} \cdot d\mathbf{a}\right)$$

integrating over *all* space: the surface integral goes to zero, and we are left with

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau$$

Example 9. Find the energy of a uniformly charged spherical shell of total charge *q* and radius *R*.

$$W = \frac{1}{2}\int \sigma V \, da$$

the potential at the surface of the sphere is constant; $(1/4\pi\epsilon_0)q/R$

$$\Rightarrow \quad W = \frac{1}{8\pi\epsilon_0}\frac{q}{R}\int \sigma \, da = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$$

Solution 2

Inside the sphere, **E** = **0**; outside, $\longrightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \longrightarrow E^2 = \frac{q^2}{(4\pi\epsilon_0)^2 r^4}$

$$W_{\text{tot}} = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int_{\text{outside}} \left(\frac{q^2}{r^4}\right)(r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

$$= \frac{1}{32\pi^2\epsilon_0} q^2 4\pi \int_R^\infty \frac{1}{r^2} dr = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$$

NOTE:

Because electrostatic energy is *quadratic* in the fields, it does *not* obey a superposition principle.

$$W_{\text{tot}} = \frac{\epsilon_0}{2}\int E^2\,d\tau = \frac{\epsilon_0}{2}\int (\mathbf{E}_1 + \mathbf{E}_2)^2\,d\tau$$

$$= \frac{\epsilon_0}{2}\int \left(E_1^2 + E_2^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2\right)\,d\tau$$

$$= W_1 + W_2 + \epsilon_0\int \mathbf{E}_1\cdot\mathbf{E}_2\,d\tau.$$

E.g.: If you double the charge everywhere, you *quadruple* the total energy.