THE BIOT-SAVART LAW

Steady Currents; continuous flow that has been going on forever, without change and without charge piling up anywhere.

$\longrightarrow$ produce magnetic fields that are constant in time;

Stationary charges $\Rightarrow$ **constant electric fields: electrostatics.**

Steady currents $\Rightarrow$ **constant magnetic fields: magnetostatics.**

In magnetostatics (and electrostatic) régime;

$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0}$$

$$\nabla \cdot \mathbf{J} = 0$$

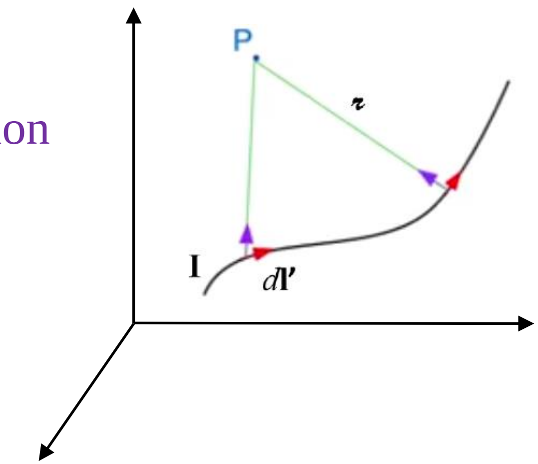
The Magnetic Field of a Steady Current (line current)

Length segment

Radial direction

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Distance



- μ_0 is the permeability of free space $\qquad \mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$

- The unit of magnetic field is Newton per Ampere-meter, or Tesla $\quad 1\ \mathrm{T} = 1\ \mathrm{N/(A\cdot m)}$

  1 Tesla = 10^4 gauss

- Biot-Savart law for magnetic field is analogous to Coulomb’s law for electric field

The magnetic field produced by a surface current:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$$

Figure

The magnetic field produced by a volume current;

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

Figure

Example 5. Find the magnetic field a distance s from a long straight wire carrying a steady current I .

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

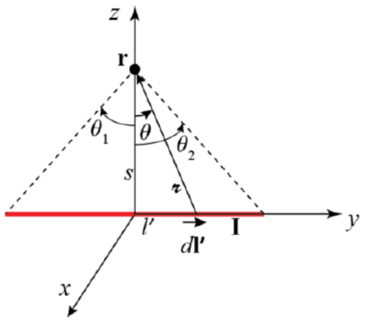
$$\boxed{\mathbf{B}(\mathbf{r}) = B\,\hat{\mathbf{x}}}$$

$$(d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}) = dl'\,|\hat{\boldsymbol{\mathscr{r}}}|\sin\alpha \longrightarrow dl'\sin\alpha = dl'\cos\theta$$

Figure

$$l' = s\tan\theta \longrightarrow dl' = \frac{s}{\cos^2\theta}\,d\theta$$

$$s = \mathscr{r}\cos\theta \longrightarrow \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$



Figure

$$\longrightarrow \quad B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$

In the case of an *infinite* wire, $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$,

$$\boxed{B = \frac{\mu_0 I}{2\pi s}}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi s}\hat{\mathbf{x}}$$

Figure

In the region *below* the wire, **B** points *into* the page, and in general, it “circles around” the wire,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$$

Here, the field is best represented in the cylindrical coordinate.

E.g.: Find the force of attraction between two long, parallel wires a distance d apart, carrying currents I_1 and I_2 .

The field at (2) due to (1) points into the page;

Figure 20

$$B = \frac{\mu_0 I_1}{2\pi d}$$

The magnetic force on a segment of current-carrying wire is:

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{I} \times \mathbf{B})\, dl = I \int (d\mathbf{l} \times \mathbf{B})$$

The Lorentz force law predicts a force directed towards (1), of magnitude:

$$F = I_2 \left(\frac{\mu_0 I_1}{2\pi d} \right) \int dl$$

The *total* force is infinite (for infinite wire), but the force per unit length is;

$$f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$

Note: If the currents are antiparallel (one up, one down), the force is repulsive.

Example 6. Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I .

The line element $d\mathbf{l}'$ produces the field $d\mathbf{B}$ at $\mathbf{r}$.
The horizontal components of this field cancels out.

Figure

The vertical component of this field is

$$dB(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{dl'}{\mathscr{r}^2}\cos\theta$$

The total field, which is in the z-direction, is;

$$\mathbf{B}(z) = \frac{\mu_0 I}{4\pi}\int\frac{\cos\theta}{\mathscr{r}^2}dl' = \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{\mathscr{r}^2}\int dl' = \frac{\mu_0 I}{4\pi}\frac{R}{\mathscr{r}^3}(2\pi R) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

(Notice that $d\mathbf{l}'$ and $\boldsymbol{\mathscr{r}}$ are perpendicular, in this case; the factor of $\cos\theta$ projects out the vertical component.)
$\cos\theta$ and $\mathscr{r}^2$ are constants, and $\int dl'$ is simply the circumference, $2\pi R$.

What is the field at the center?

$$\mathbf{B}(0) = \frac{\mu_0 I}{2}\frac{R^2}{R^3} = \frac{\mu_0 I}{2R}$$

THE DIVERGENCE AND CURL OF **B**

What is the divergence of **B**? $\nabla \cdot \mathbf{B} = 0$

What is the curl of **B? Should be $\nabla \times \mathbf{B} \neq \mathbf{0}$**

A straight wire with current I;

$$\mathrm{B}(s) = \frac{\mu_0 I}{2\pi s}$$

Figure

For circular path of radius s

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$

The line integral is independent of ***s***

because ***B*** *de*creases at the same rate as the circumference *in*creases.

Figure 27

In fact, it doesn’t have to be a circle; *any* closed loop that encloses the wire would give the same answer.

For an arbitrary path enclosing the current carrying wire
The field is best represented in the cylindrical coordinate

Figure

use cylindrical coordinates (s, φ, z),
with the current flowing along the z axis;

$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad d\mathbf{l} = ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \cdot (ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}) = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I$$

Hence, if there is more than a wire, each wire that passes through the loop contributes $\mu_0 I$:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$$

Figure

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_1 + \mu_0 I_2 + \mu_0 I_3 + \cdots = \mu_0 I_{enc}$$

I_{enc} stands for the total current enclosed by the integration path.

If the flow of charge is represented by a volume current density $\mathbf{J}$;

$$I_{enc} = \int \mathbf{J} \cdot d\mathbf{a} \quad \Longrightarrow \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

Stokes' theorem;

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Ampère's Law

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \quad \Longrightarrow \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

(If there is no symmetry, one has to use Biot-Savart law to calculate the magnetic field.)

Applications of Ampère's Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \longrightarrow \text{(Ampere's law in differential form)}$$

Figure

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \longrightarrow \text{(Ampere's law in integral form)}$$

Electrostatics : Coulomb $\longrightarrow$ Gauss,

Magnetostatics : Biot−Savart $\longrightarrow$ Ampère

Example 7. Find the magnetic field a distance *s* from a long straight wire, carrying a steady current *I*

Make an Amperian loop of radius *s* enclosing the current.

Figure

Since it is an infinite wire, the magnetic field must be circularly symmetric. Therefore,

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi s}$$

Example 9. Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R, each carrying a steady current I.

Figure

Figure

$$\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi (2\pi s) = \mu_0 I_{\text{enc}} = 0$$

$$B_\varphi = 0$$

For an Amperian loop outside;

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \Rightarrow \quad B(a)l - B(b)l = 0 \quad \Rightarrow \quad B(a) = \mathrm{B}(b)$$

$$\text{But since} \quad B(\infty) = 0, \quad B(a) = \mathrm{B}(b) = \mathrm{B}(\infty) = 0$$

Figure

For an Amperian loop inside

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \Rightarrow \quad B(s)l - 0 = \mu_0 n I l \quad \Rightarrow \quad B(s) = \mu_0 n I$$

$$\left.\begin{aligned} &\mathbf{B} = 0 && \text{for } (s > R) \\ &\mathbf{B} = \mu_0 n I \hat{\mathbf{z}} && \text{for } (s < R) \end{aligned}\right\}$$

the field inside is *uniform*—it doesn't depend on the distance from the axis.