Linear and Time Invariant Systems

Lecture 8

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Recap:

We have seen systems with:

- memory
- invertibility
- causality
- stability
- linearity
- time invariance

In this lecture, we focus on LTI (Linear Time Invariant) systems.

Linearity and Time Invariance:

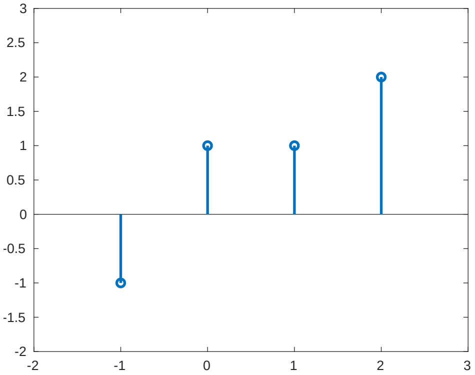
Time Invariance: A time invariant system is not affected from the time origin of the input.

Linearity: if a system is linear, linear combination of inputs will produce linear combination of outputs from each individual input:

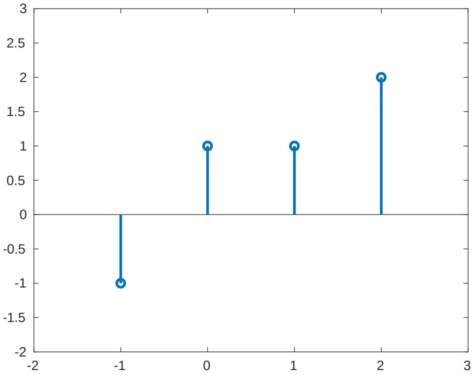
$$x_1[n] \rightarrow y_1[n]$$
$$x_2[n] \rightarrow y_2[n]$$
$$\alpha x_1[n] + \beta x_2[n] \rightarrow \alpha y_1[n] + \beta y_2[n]$$

The Strategy to Exploit Properties of LTI Systems:

We will decompose an inputs into their simple components
For example, let $x[n]$ be a discrete time system:



$x[n]$ can be written as:



$$
\begin{aligned}
x[n] &= x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] \\
&= -\delta[n+1] + \delta[n] + \delta[n-1] + 2\delta[n-2]
\end{aligned}
$$

Generalization:

For discrete time signals:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

For continuous time signals:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The Main Intuition:

Linearity enables us to think the total response of the system is equivalent to the sum of responses to each individual components of the input.

Since the system is time invariant, if the input is shifted by n_0 the output will be the same except it will be shifted by the same amount n_0.

Impulse Response:

The system's response (output) when the input is an impulse ($\delta[n]/\delta(t)$) is called the impulse response of that system. Impulse response is usually denoted with $h[n]/h(t)$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

This operation is called convolution and denoted as:

$$y[n] = x[n] * h[n] \qquad y(t) = x(t) * h(t)$$

Impulse Response

We are exploring the properties of LTI systems because such systems can be represented by their impulse responses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

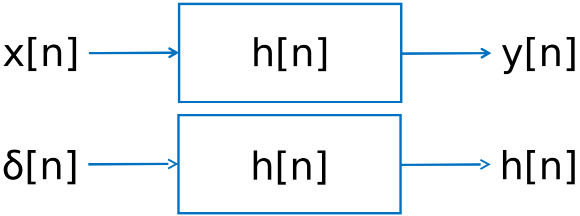
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

For continuous time signals:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

LTI Systems, Their Impulse Responses and Convolution



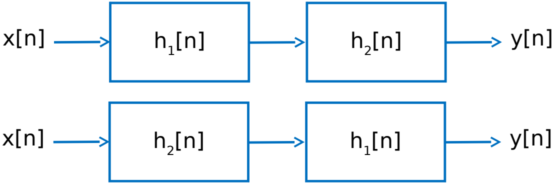
$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Properties of LTI Systems:

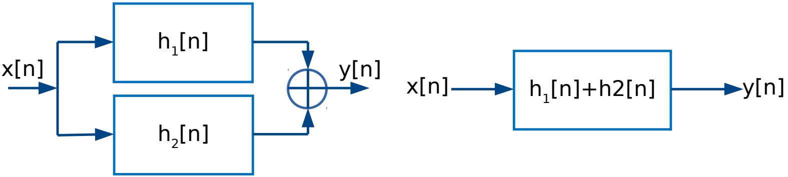
- Commutative
- Distributive
- Associative
- Memory
- Invertibility
- Causality
- Stability

Commutative:



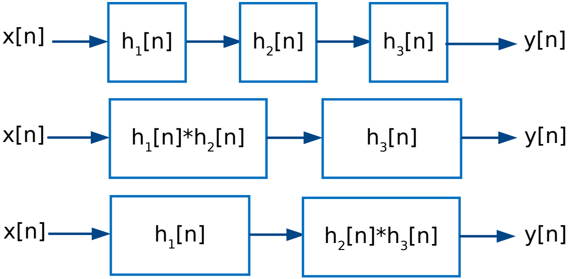
$$y[n] = (x[n] * h_1[n]) * h_2[n]$$
$$y[n] = (x[n] * h_2[n]) * h_1[n]$$
$$y[n] = x[n] * (h_1[n] * h_2[n])$$

Distributive:



$$y[n] = x[n] * h_1[n] + x[n] * h_2[n]$$
$$= x[n] * (h_1[n] + h_2[n])$$

Associative:



$$x[n] * (h_1[n] * h_2[n]) * h_3[n] = x[n] * h_1[n] * (h_2[n] * h_3[n])$$

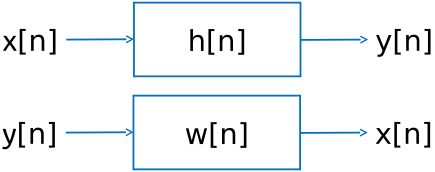
System with Memory

A memoryless system must not use past and anticipate about the future values of its input.

Is the following system with memory?

$$y[n] = 3x[n] + 2x[n+1]$$

Invertible System:



$w[n]$ is the inverse response of $h[n]$. Remember: If a system is invertible, different inputs should lead to unique outputs.

Causality:

In order for a system to be causal, the system should not anticipate about future values of the input.

Is the following system causal?

$$y[n] = \sum_{-\infty}^{n} x[k]$$

Stability:

For stable systems, bounded input should produce bounded output. This is also called BIBO stable.

Is the following system causal?

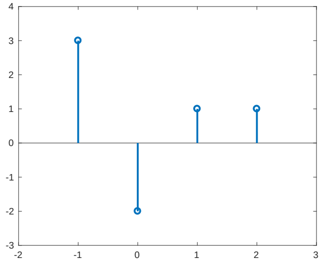
$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

Example:

For the following system, sketch its impulse response, $h[n]$.

$$y[n] = 3x[n+1] - 2x[n] + x[n-1] + x[n-2]$$

Answer:



$$h[n] = 3\delta[n+1] - 2\delta[n] + \delta[n-1] + \delta[n-2]$$

Example:

Given the following system:

$$y[n] = \alpha^n x[n], \alpha > 0$$

Determine whether the system is:

- Memoryless
- Causal
- Linear
- Time Invariant