Continuous-Time Fourier Series

Lecture 10

Dr. Görkem Saygılı

Department of Biomedical Engineering
Ankara University

Signals & Systems, 2018-2019 Fall

Content of This Lecture:

In this lecture, we will learn how to represent continuous time periodic signals with complex exponentials.

Later, we will learn how to extend this to aperiodic signals (Fourier Transform).

Response of LTI to Complex Exponentials:

If we can represent a signal as a linear combination of basic signals then the response of an LTI system to this signal will be the linear combination of responses to each of these basic signals.

For LTI systems these basic signals are complex exponentials (for continuous time: e^{st}, for discrete time: z^n). Because:

- Complex exponentials exit an LTI system with only a change (if any) in their amplitude.
- Broad classes of signals can be represented with linear combination of complex exponentials.

Eigenfunctions of LTI Systems:

By definition, an eigenfunction of a system is a signal such that the response of the system to that signal is just the signal itself multiplied with a constant.

Complex exponentials are eigenfunctions of LTI systems.

They enter an LTI system and leave the system as they are but just multiplied by a constant.

Periodic Signals:

Remember, if signal $x(t)$ is periodic with period T, then $x(t) = x(t+T)$.

- Fundamental period: T_0 where $T = kT$ and $k \in Z$ (sec).
- Fundamental frequency: $\omega_0 = 2\pi f_0$ (radians/sec).

Harmonically-Related Complex Exponentials:

$$\sigma_k(t) = e^{jk\omega_0 t}, k \in Z \quad (1)$$

where ω_0 is the fundamental frequency.

$\omega_0 = 2\pi f_0$ and $f_0 = \frac{1}{T_0}$

The common period of this set of signals is T_0.

Representing A Periodic Signal with Complex Exponentials:

We can represent a continuous time periodic signal $x(t)$ with harmonically related complex exponentials as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{2}$$

This representation is called Fourier Series representation of $x(t)$ which is a continuous time periodic signal with a fundamental period of T_0.

Important Remarks:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- In this representation, a_k's are called Fourier series coefficients (or spectral coefficients)
- Almost all periodic continuous time signals that are common in engineering can be written in this form.
- If a continuous time periodic signal can be written in this form, then it has to be periodic with the fundamental period of $T_0 = \frac{2\pi}{\omega_0}$

Finding Fourier Series Coefficients (a_k's):

We can calculate a_k's by evaluating the following integral:

$$a_k = \frac{1}{T_0} \int_0^T x(t) e^{-jk\omega_0 t} \tag{3}$$

The boundary encapsulates any period of the signal. Hence, it may be considered from 0 to T_0 or $-\frac{T_0}{2}$ to $\frac{T_0}{2}$.

Properties of Fourier Series Coefficients - 1:

If a CT periodic signal satisfies:

$$x(t) = x^*(t) \tag{4}$$

Hence $x(t)$ is a real CT periodic signal. Then,

$$a_k = a_k^* \tag{5}$$

Properties of Fourier Series Coefficients - 2:

If $x(t)$ is even:

$$x(t) = x(-t) \Rightarrow a_k = a_{-k} \tag{6}$$

If $x(t)$ is odd:

$$x(t) = -x(-t) \Rightarrow a_k = -a_{-k} \tag{7}$$

Properties of Fourier Series Coefficients - 3:

$$x(t) \leftrightarrow a_k \Rightarrow x(t - t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0} \quad (8)$$

$$x(t) \leftrightarrow a_k \Rightarrow x(-t) \leftrightarrow a_{-k} \quad (9)$$

$$x(t) \leftrightarrow a_k \Rightarrow \frac{d}{dt}x(t) \leftrightarrow (jk\omega_0)a_k \quad (10)$$

Response of LTI Systems:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{11}$$

$$y(t) = x(t) * h(t), \tag{12}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} \tag{13}$$

Linear Combination of Two CT Periodic Signals:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad (14)$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}, \quad (15)$$

$$Ax(t) + By(t) = \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k) e^{jk\omega_0 t} \quad (16)$$