CHAPTER 3. APPLICATIONS of FIRST ORDER DIFFERENTIAL EQUATIONS

3.1. Geometrical Problems

Example 1) Find all plane curves for which every slope of tangent is equal to ordinate at that point.

Solution. Since every slope of tangent is equal to ordinate at that point, we have the following seperable equation

$$\frac{dy}{dx} = y$$

Integrating this equation we obtain the plane curves

$$\ln y = x + \ln c \text{ or } y = ce^x.$$

Example 2) Find the plane curves which intersects the x-axis at the point 2 with the slope of tangent is equal to xe^{-y}.

Solution. Now, we have the following differential equation

$$\frac{dy}{dx} = xe^{-y}$$

Integrating this equation we obtain the plane curves

$$e^y = \frac{x^2}{2} + c$$

Applying the condition $y(2) = 0$, we get $c = -1$ and

$$y = \ln\left(\frac{x^2}{2} - 1\right).$$

3.2. Orthogonal and Oblique Trajectories

Definition. Let

$$F(x, y, c) = 0 \tag{1}$$

be a given one-parameter family of curves in the xy-plane. A curve that intersects the curves of the family (1) at right angles is called an orthogonal trajectory of the given family.

Method. *Step 1.* To find the orthogonal trajectories of a family of curves (1), first differentiate equation (1) implicitly with respect to x and obtain the differential equation of the given family of curves.

Step 2. Eliminate the parameter c between the derived equation and the given equation (1).

Step 3. Let us assume that the resulting differential equation of the family (1) can be expressed in the form

$$\frac{dy}{dx} = f(x, y)$$

Step 4. Since an orthogonal trajectory of the given family intersects each curve of given family atright angels, the slope of the orthogonal trajectory to γ at (x, y) is $-\dfrac{1}{f(x, y)}$. So, the differential equation of the family of orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

Example 1) Find the orthogonal trajectories of the family of parabola $y = cx^2$, where c is an arbitrary constant.

Solution. Differentiating $y = cx^2$, we obtain the differential equation

$$\frac{dy}{dx} = 2cx. \tag{2}$$

Substituting $c = \dfrac{y}{x^2}$ into (2) we obtain

$$\frac{dy}{dx} = \frac{2y}{x}$$

which is the differential equation of the given family of parabolas. So,

$$\frac{dy}{dx} = \frac{-x}{2y} \tag{3}$$

is the differential equation of the orthogonal trajectories of the family $y = cx^2$. Solving (3) by seperating variables, we obtain

$$2y^2 + x^2 = c^2,$$

where c is a constant.

Example 2) Find the orthogonal trajectories of the family $y = \dfrac{cx}{1+x}$.

Example 3) Find the orthogonal trajectory that passes through the point $(1, 2)$ of the family $x^2 + y^2 = cy$.

Definition. Let $F(x, y, c) = 0$ be a one parameter family of curves. A curve which intersects the curves of the given family at a constant angle $\alpha \neq 90^\circ$ is called an oblique trajectory of the given family.

Suppose the differential equation of the given family is

$$\frac{dy}{dx} = f(x, y).$$

Then the differential equation of a family of oblique trajectories is given by

$$\frac{dy}{dx} = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}.$$

in the variables X and Y.

Example 4) Find the family of oblique trajectories that intersect the family of circles $x^2 + y^2 = c^2$ at angle $45°$.

Solution. From $x^2 + y^2 = c^2$ we get $2x + 2y\dfrac{dy}{dx} = 0$ or $\frac{dy}{dx} = \dfrac{-x}{y}$. So, $f(x, y) = \dfrac{-x}{y}$ and the differential equation of the family of oblique trajectories is

$$\frac{dy}{dx} = \frac{-\frac{x}{y} + 1}{1 + \frac{x}{y}} = \frac{y - x}{y + x} \tag{4}$$

It is clear that equation (4) is a homogeneous differential equation. Applying the transformation $y = xv,\ v = v(x)$, we obtain the following seperable differential equation

$$x\frac{dv}{dx} + v = \frac{x(v-1)}{x(v+1)}$$

or

$$\frac{v+1}{v^2+1}dv = -\frac{dx}{x}$$

Integrating last equation we obtain the solution of seperable equation as

$$\frac{1}{2}\ln(v^2 + 1) + \arctan v = -\ln|x| - \ln c$$

or

$$\ln(v^2 + 1)c^2x^2 + 2\arctan v = 0.$$

Since $y = vx$, the solution of homogeneous equation is

$$\ln c^2(x^2 + y^2) + 2\arctan\frac{y}{x} = 0.$$

Example 5) Find the family of oblique trajectories that intersect the family of lines $y = cx$ at angle $45°$.

Example 6) Find the family of oblique trajectories that intersect the family of curves $(x + c)y = 1$ at angle $\alpha = \arctan 4$.