Calculus II
Week 10 Lecture

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Series: Geometric series

A geometric series is any series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or, with an index shift the geometric series will often be written as,

$$\sum_{n=0}^{\infty} ar^{n}$$

Series: Geometric series

If $|r| < 1$ then

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n} = \frac{a}{1-r}$$

Example: Determine if the following series converge or diverge.

$$\sum_{n=0}^{\infty} \frac{2^n}{5^{n-1}}$$

Series:Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called Harmonic series.
The harmonic series is divergent and we will show that later.

Series:Integral Test

Suppose that $f(x)$ is a continuous, positive and decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$ then,

If $\displaystyle\int_k^{\infty} f(x)\,dx$ is convergent so is $\displaystyle\sum_{n=k}^{\infty} a_n$

If $\displaystyle\int_k^{\infty} f(x)\,dx$ is divergent so is $\displaystyle\sum_{n=k}^{\infty} a_n$

Series:Harmonic series

Consider the function $f(x) = \dfrac{1}{x}$ on the interval $[1, \infty)$. The area under the curve is approximately,

$$
\begin{aligned}
A &\approx \left(\frac{1}{1}\right)(1) + \left(\frac{1}{2}\right)(1) + \left(\frac{1}{3}\right)(1) + \left(\frac{1}{4}\right)(1) + \left(\frac{1}{5}\right)(1) + \cdots \\
&= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{1}{n}
\end{aligned}
$$

This implies that $A \approx \sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\, dx = \infty$. Thus harmonic series is divergent.

Series:Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ for all n. Then
If $\sum b_n$ is convergent then so is $\sum a_n$.
If $\sum a_n$ is divergent then so is s $\sum b_n$.

Example

Determine if the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - 1\,(n)}$$

Series:Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$. Define,

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$

If c is positive and finite then either both series converge or both series diverge.

Example

Determine if the following series is convergent or divergent

$$\sum_{n=0}^{\infty} \frac{1}{2^n - n}$$

Series:Alternating Test

The test that we are going to look into in this section will be a test for alternating series. An alternating series is any series, $\sum a_n$ for which the series terms can be written in one of the following two forms.

$$a_n = (-1)^n b_n \qquad b_n \geq 0$$
$$a_n = (-1)^{n+1} b_n \qquad b_n \geq 0$$

Series:Alternating Test

Suppose we have an alternating series $\sum (-1)^n b_n$ where $b_n \geq 0$. If $\lim_{n \to \infty} b_n = 0$ and $\{b_n\}$ is a decreasing sequence then $\sum (-1)^n b_n$ is convergent.

Example

Determine if the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Series:Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,
if $L < 1$ the series is absolutely convergent (and hence convergent).
if $L > 1$ the series is divergent.
if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Series:Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,
if $L < 1$ the series is absolutely convergent (and hence convergent).
if $L > 1$ the series is divergent.
if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Example

Determine if the following series is convergent or divergent

$$\sum_{n=0}^{\infty} \frac{n!}{3^n}$$