Calculus II
Week 11 Lecture

Oktay Olmez and Serhan Varma

Power series

A power series about a, or just power series, is any series that can be written in the form,

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Power series

$a - R < x < a + R$	power series converges
$x < a - R$ and $x > a + R$	power series diverges

R is called is called the radius of convergence.
The interval of all xs, including the endpoints if need be, for which the power series converges is called the interval of convergence of the series.
This interval must contain $a - R < x < a + R$.

Example

Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n$$

Power series representation

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{provided } |x| < 1 \tag{1}$$

Example: Find a power series representation for the following function and determine its interval of convergence.

$$g(x) = \frac{1}{1+x^3}$$

Power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

Derivative of power series representation

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

Integral of power series representation

$$\begin{aligned}\int f(x)\, dx &= \int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx \\ &= \sum_{n=0}^{\infty} \int c_n (x-a)^n \, dx \\ &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\end{aligned}$$

Fact

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence of R $\int f(x)\, dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx$ and $f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n$ both have the radius R.

Fact

Find a power series representation for the following function and determine its interval of convergence.

$$h(x) = \ln(2 - x)$$

Fact

Find a power series representation for the following function and determine its interval of convergence.

$$h(x) = \frac{1}{1 + x^2}$$