Abstract Mathematics
Lecture 12

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We will discuss how to show that

1. an object is an element of a set,
2. how to prove one set is a subset of another and
3. how to prove two sets are equal.

Proofs Involving Sets

Remember that;

$$\begin{aligned} A \times B &= \{(x,y) : x \in A, y \in B\}, \\ A \cup B &= \{x : (x \in A) \vee (x \in B)\}, \\ A \cap B &= \{x : (x \in A) \wedge (x \in B)\}, \\ A - B &= \{x : (x \in A) \wedge (x \notin B)\}, \\ \overline{A} &= U - A, \end{aligned}$$

How to Prove $a \in A$

To show that $a \in \{x : P(x)\}$ we just need to show that $P(a)$ is true.

Likewise, to show $a \in \{x \in S : P(x)\}$, we need to confirm that $a \in S$ and that $P(a)$ is true.

Proofs Involving Sets

Example

Lets investigate elements of $A = \{x : x \in \mathbb{N}$ *ve* $7|x\}$. *This set has form* $A = \{x : P(x)\}$ *where* $P(x)$ *is the open sentence* $(x \in \mathbb{N}) \wedge (7|x)$.

Thus $21 \in A$ *because* $P(21)$ *is true. Similarly,* $7, 14, 28, 35, etc.,$ *are all elements of* A.

But $8 \notin A$ *because P(8) is false.*

Example

Consider the set $A = \{X \in \mathcal{P}(\mathbb{N}) : |X| = 3\}$. *We know that* $\{4, 13, 45\} \in A$ *because* $\{4, 13, 45\} \in \mathcal{P}(\mathbb{N})$ *and* $\left|\{4, 13, 45\}\right| = 3$. *But*

$\{1, 2, 3, 4\} \notin A$ *since* $\left|\{1, 2, 3, 4\}\right| \neq 3$.

How to Prove $A \subseteq B$

To prove that $A \subseteq B$, we just need to prove that the conditional statement If $a \in A$, then $a \in B$ is true.

This can be proved directly, by assuming $a \in A$ and deducing $a \in B$.

The contrapositive approach is another option: Assume $a \notin B$ and deduce $a \notin A$.

Proofs Involving Sets

Example

Prove that $\{x \in \mathbb{Z} : 18|x\} \subseteq \{x \in \mathbb{Z} : 6|x\}$.

Proof.

Suppose $a \in \{x \in \mathbb{Z} : 18|x\}$. This means that $a \in \mathbb{Z}$ and $18|a$.

By definition of divisibility, there is an integer c for which $a = 18c$.

Consequently $a = 6(3c)$, and from this we deduce that $6|a$. Therefore a is one of the integers that 6 divides, so $a \in \{x \in \mathbb{Z} : 6|x\}$.

Weve shown $a \in \{x \in \mathbb{Z} : 18|x\}$ implies $a \in \{x \in \mathbb{Z} : 6|x\}$, so it follows that $\{x \in \mathbb{Z} : 18|x\} \subseteq \{x \in \mathbb{Z} : 6|x\}$. □

Example

Prove that if A and B are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Suppose $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. By definition of union, this means $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$.
Therefore $X \subseteq A$ or $X \subseteq B$ (by definition of power sets). We consider cases.

Case 1. Suppose $X \subseteq A$. Then $X \subseteq A \cup B$, and this means $X \in \mathcal{P}(A \cup B)$.
Case 2. Suppose $X \subseteq B$. Then $X \subseteq A \cup B$, and this means $X \in \mathcal{P}(A \cup B)$.

The above cases show that $X \subseteq \mathcal{P}(A \cup B)$. Thus weve shown that $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$ implies $X \in \mathcal{P}(A \cup B)$, and this completes the proof that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.