Abstract Mathematics
Lecture 12

Oktay Ölmez

Proofs Involving Sets

How to Prove $A = B$

The standard procedure for proving $A = B$ is: Prove both $A \subseteq B$ and $B \subseteq A$.

Sometimes, you can prove two sets are equal by working out a series of equalities leading from one set to the other.

Example

Given sets A, B, and C, prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof.

Just observe the following sequence of equalities.

$$\begin{aligned}
A \times (B \cap C) &= \big\{(x,y) : (x \in A) \wedge (y \in B \cap C)\big\} \\
&= \big\{(x,y) : (x \in A) \wedge (y \in B) \wedge (y \in C)\big\} \\
&= \big\{(x,y) : (x \in A) \wedge (x \in A) \wedge (y \in B) \wedge (y \in C)\big\} \\
&= \big\{(x,y) : ((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \in C))\big\} \\
&= \big\{(x,y) : (x \in A) \wedge (y \in B)\big\} \cap \\
&\quad\ \big\{(x,y) : (x \in A) \wedge (y \in C)\big\} \\
&= (A \times B) \cap (A \times C)
\end{aligned}$$

This completes the proof. □

Perfect Numbers

Definition

A number $p \in \mathbb{N}$ is perfect if it equals the sum of its positive divisors less than itself. Some examples follow.

- *The number 6 is perfect since* $6 = 1 + 2 + 3$.
- *The number 28 is perfect since* $28 = 1 + 2 + 4 + 7 + 14$.
- *The number 496 is perfect since* $496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$.

Proofs Involving Sets

Prove the theorem:

Example

If $A = \left\{ 2^{n-1}(2^n - 1) : n \in \mathbb{N} \text{ ve } 2^n - 1 \text{ is prime number} \right\}$ *and* $P = \left\{ p \in \mathbb{N} : p \text{ is perfect number} \right\}$ *then* $A \subseteq P$.

Assume A and P are as stated. Suppose $p \in A$. By definition of A, this means:

$$p = 2^{n-1}(2^n - 1)$$

for some $n \in \mathbb{N}$ for which $2^n - 1$ is prime.

We want to show that p 2 P, that is, we want to show p is perfect.

Notice that since $2^n - 1$ is prime, any divisor of $2^{n-1}(2^n - 1)$ must have the form 2^k or $2^k(2^n - 1)$ for $0 \leq k \leq n - 1$.

Proofs Involving Sets

The positive divisors of p are as follows:

$$\begin{array}{llllll} 2^0, & 2^1, & 2^2, & \dots & 2^{n-2}, & 2^{n-1}, \\ 2^0(2^n-1), & 2^1(2^n-1), & 2^2(2^n-1), & \dots & 2^{n-2}(2^n-1), & 2^{n-1}(2^n-1). \end{array}$$

If we add up all these divisors except for the last one (which equals p) we get the following:

$$\begin{aligned}
\sum_{k=0}^{n-1} 2^k + \sum_{k=0}^{n-2} 2^k(2^n-1) &= \sum_{k=0}^{n-1} 2^k + (2^n-1)\sum_{k=0}^{n-2} 2^k \\
&= (2^n-1) + (2^n-1)(2^{n-1}-1) \\
&= \left[1 + (2^{n-1}-1)\right](2^n-1) \\
&= 2^{n-1}(2^n-1) \\
&= p
\end{aligned}$$

Therefore p is perfect, by definition of a perfect number. Thus $p \in P$.

Proofs Involving Sets

1. Prove that $\{12n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.
2. Prove that $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.
3. If $k \in \mathbb{Z}$, then $\{n \in \mathbb{Z} : n \mid k\} \subseteq \{n \in \mathbb{Z} : n \mid k^2\}$.
4. If $m, n \in \mathbb{Z}$, then $\{x \in \mathbb{Z} : mn \mid x\} \subseteq \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.
5. If p and q are positive integers, then $\{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\} \neq \emptyset$.
6. Suppose A, B and C are sets. Prove that if $A \subseteq B$, then $A - C \subseteq B - C$.
7. Suppose A, B and C are sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$.

Proofs Involving Sets

8. If A, B and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

9. If A, B and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

10. If A and B are sets in a universal set U, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

11. If A and B are sets in a universal set U, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

12. If A, B and C are sets, then $A - (B \cap C) = (A - B) \cup (A - C)$.

13. If A, B and C are sets, then $A - (B \cup C) = (A - B) \cap (A - C)$.

14. If A, B and C are sets, then $(A \cup B) - C = (A - C) \cup (B - C)$.

15. If A,B and C are sets, then $(A \cap B) - C = (A - C) \cap (B - C)$.

16. If A,B and C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

17. If A,B and C are sets, then $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

18. If A,B and C are sets, then $A \times (B - C) = (A \times B) - (A \times C)$.

19. Prove that $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$, but $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$

20. Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

21. Suppose A and B are sets. Prove $A \subseteq B$ if and only if $A - B = \emptyset$.

22. Let A and B be sets. Prove that $A \subseteq B$ if and only if $A \cap B = A$.