Abstract Mathematics
Lecture 13

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Disproof

We have dealt with one major theme: Given a statement, prove that is it true.

Have you ever wondered what would happen if you were given a false statement to prove?

The answer is that no (correct) proof would be possible, for if it were, the statement would be true, not false.

Disproof

But how would you convince someone that a statement is false?

It turns out that there is a very simple and utterly convincing procedure that proves a statement is false. The process of carrying out this procedure is called **disproof**.

Disproof

Mathematicians have a special name for the statements in the category that they suspect (but havent yet proved) are true. Such statements are called conjectures.

Three Types of Statements:

1. Known to be true
   - a Pythagorean theorem
   - b Fermats last theorem
2. Truth unknown
   - a All perfect numbers are even.
   - b Any even number greater than 2 is the sum of two primes. (Goldbachs conjecture)
3. Known to be false
   - a All prime numbers are odd.
   - b Some quadratic equations have three solutions.

Disproof

Suppose you want to disprove a statement P. The way to do this is to prove that $\sim P$ is true.

How to disprove P: **Prove** $\sim P$.

Disproof

Disproving Universal Statements: Counterexamples

How to disprove $\forall x \in S, P(x)$

Produce an example of an $x \in S$ that makes $P(x)$ false.

Things are just as simple if we want to disprove a conditional statement $P(x) \rightarrow Q(x)$.

How to disprove $P(x) \rightarrow Q(x)$

Produce an example of an x that makes $P(x)$ true and $Q(x)$ false.

Disproof

There is a special name for an example that disproves a statement: It is called a counterexample.

As our first example, we will work through the process of deciding whether or not the following conjecture is true.

Disproof

Example

For every $n \in \mathbb{Z}$*, the integer* $f(n) = n^2 - n + 11$ *is prime.*

In resolving the truth or falsity of a conjecture, its a good idea to gather as much information about the conjecture as possible.

Disproof

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	23	17	13	11	11	13	17	23	31	41	53	67	83	101

In every case, f (n) is prime, so you may begin to suspect that the conjecture is true.

Disproof

Before attempting a proof, lets try one more n. Unfortunately, $f(11) = 11^2 - 11 + 11 = 11^2$ is not prime.

The conjecture is false because $n = 11$ is a counterexample.

Disproof

Example

Either prove or disprove the following conjecture.

Conjecture

If A, B and C are sets, then $A - (B \cap C) = (A - B) \cap (A - C)$.

Proof.

This conjecture is false because of the following counterexample. Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$ and $C = \{2, 3\}$. Notice that $A - (B \cap C) = \{1, 3\}$ and $(A - B) \cap (A - C) = \emptyset$, so $A - (B \cap C) \neq (A - B) \cap (A - C)$. □

Disproof

Disproving Existence Statements

Example

Either prove or disprove the following conjecture.

Conjecture

There is a real number x for which $x^4 < x < x^2$.

After applying some intelligent guessing to locate such an x we run into trouble.

Lets see if we can disprove it. According to our strategy for disproof, to disprove it we must prove its negation.

$$\sim\left(\exists x \in \mathbb{R}, x^4 < x < x^2\right) \quad = \quad \forall x \in \mathbb{R}, \sim\left(x^4 < x < x^2\right)$$

Disproof

This can be proved with contradiction, as follows. Suppose for the sake of contradiction that there is an x for which $x^4 < x < x^2$.

Then x must be positive since its greater than the non-negative number x^4

Dividing all parts by x and subtracting 1 we get $x^3 - 1 < 0 < x - 1$

Disproof

$$\begin{aligned} x^3 - 1 &< 0 < x - 1 \\ (x-1)(x^2+x+1) &< 0 < (x-1) \\ x^2+x+1 &< 0 < 1 \end{aligned}$$

This is a contradiction because x being positive forces $x^2 + x + 1 < 0$.

Disproof by Contradiction

How to disprove $P(x)$ with contradiction:

Assume P is true, and deduce a contradiction.

Disproof

Example

Disprove the following conjecture.

Conjecture

There is a real number x for which $x^4 < x < x^2$.

Suppose for the sake of contradiction that this conjecture is true.

Let x be a real number for which $x^4 < x < x^2$. Then x is positive, since it is greater than the non-negative number x^4

Dividing all parts by x and subtracting 1 we get $x^3 - 1 < 0 < x - 1$

Disproof

$$\begin{aligned} x^3 - 1 &< 0 < x - 1 \\ (x-1)(x^2+x+1) &< 0 < (x-1) \\ x^2 + x + 1 &< 0 < 1 \end{aligned}$$

This is a contradiction because x being positive forces $x^2 + x + 1 < 0$.

Exercises

Each of the following statements is either true or false. If a statement is true, prove it. If a statement is false, disprove it.

1. If $x, y \in \mathbb{R}$, then $|x+y| = |x| + |y|$.
2. If $n \in \mathbb{Z}$ and $n^5 - n$ is even, then n is even.
3. Suppose $a, b \in \mathbb{Z}$. If $a|b$ and $b|a$ then $a = b$.