Abstract Mathematics
Lecture 14

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Mathematical Induction

Consider this statement:

Conjecture

The sum of the first n odd natural numbers equals n^2.

How to prove it?

Mathematical Induction

n	sum of the first n odd natural numbers	n^2
1	$1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	1
2	$1 + 3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	4
3	$1 + 3 + 5 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	9
4	$1 + 3 + 5 + 7 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$	16
5	$1 + 3 + 5 + 7 + 9 = \ldots\ldots\ldots\ldots\ldots\ldots$	25
$\vdots$	$\vdots$	$\vdots$
n	$1 + 3 + 5 + 7 + 9 + 11 + \cdots + (2n-1) = \ldots\ldots$	n^2
$\vdots$	$\vdots$	$\vdots$

Note that in the first five lines of the table, the sum of the first n odd numbers really does add up to n^2.

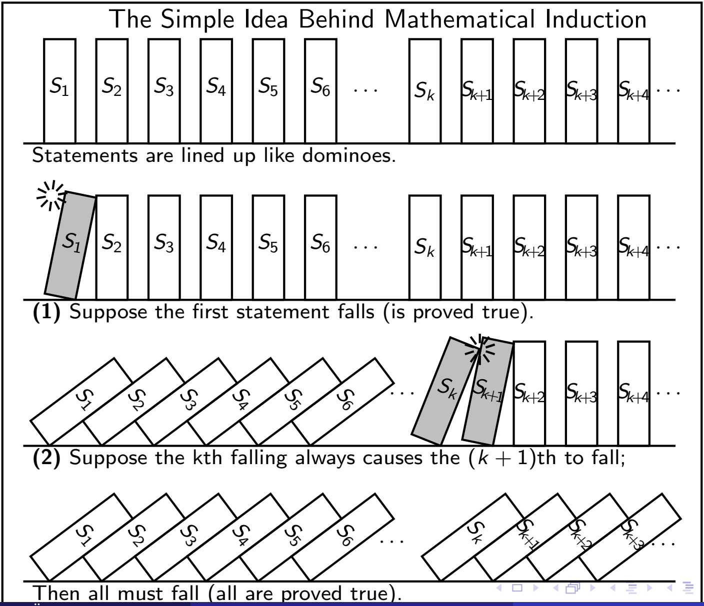
The table raises a question. Does the sum $1 + 3 + 5 + 7 + 9 + 11 + \cdots + (2n - 1)$ really always equal n^2?

Mathematical induction answers just this kind of question, where we have an infinite list of statements

Let

$$\begin{array}{l} S_1 : 1 = 1^2 \\ S_2 : 1 + 3 = 2^2 \\ S_3 : 1 + 3 + 5 = 3^2 \\ \vdots \\ S_n : 1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2 \\ \vdots \end{array}$$

Mathematical Induction



Proof by Induction

Proposition

The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

Proof.

(Induction).
(1) Prove that the first statement S_1 is true.
(2) Given any integer $k \geq 1$, prove that the statement $S_k \to S_{k+1}$ is true.
It follows by mathematical induction that every S_n is true. □

Mathematical Induction

Example

Proposition

If $n \in \mathbb{N}$, *then* $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$.

Proof.

We will prove this with mathematical induction.

1. *Observe that if* $n = 1$, *this statement is* $1^2 = 1$, *which is obviously true.*
2. *That is, we must show that if* $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$, *then* $1 + 3 + 5 + 7 + \cdots + (2(k + 1) - 1) = (k + 1)^2$.

□

Cont.

We use direct proof. Suppose $1+3+5+7+\cdots+(2k-1)=k^2$. Then

$$\begin{aligned}
1+3+5+7+\cdots\cdots\cdots\cdots\cdots+\ (2(k+1)-1) &= \\
1+3+5+7+\cdots+(2k-1)+(2(k+1)-1) &= \\
\big(1+3+5+7+\cdots+(2k-1)\big)+(2(k+1)-1) &= \\
k^2\ +(2(k+1)-1) &= k^2+2k+1 \\
&= (k+1)^2
\end{aligned}$$

Thus $1+3+5+7+\cdots+(2(k+1)-1)=(k+1)^2$. It follows by induction that $1+3+5+7+\cdots+(2n-1)=n^2$ for every $n\in\mathbb{N}$. □

Try to prove the following statement.

Proposition

If n is a non-negative integer, then $5|(n^5 - n)$.

Mathematical Induction

Sometimes in an induction proof it is hard to show that $Sk \to S_{k+1}$.

It may be easier to show some lower S_m (with $m < k$) implies S_{k+1}.

For such situations there is a slight variant of induction called strong induction.

Outline for Proof by Strong Induction

Proposition

The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

Proof.

(Strong induction).
(1) Prove the first statement S_1. (Or the first several S_n, if needed.)
(2) Given any integer $k \geq 1$, prove $(S_1 \wedge S_2 \wedge \cdots \wedge S_k) \rightarrow S_{k+1}$. □