Abstract Mathematics
Lecture 14

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Outline for Proof by Strong Induction

Proposition

The statements $S_1, S_2, S_3, S_4, \dots$ are all true.

Proof.

(Strong induction).
(1) Prove the first statement S_1. (Or the first several S_n, if needed.)
(2) Given any integer $k \geq 1$, prove $(S_1 \wedge S_2 \wedge \cdots \wedge S_k) \rightarrow S_{k+1}$. □

Example

Proposition

If $n \in \mathbb{N}$, then $12 | (n^4 - n^2)$.

Proof.

We will prove this with strong induction.

- *First note that the statement is true for the first six positive integers:*
  - *For $n = 1$, 12 divides $1^4 - 1^2 = 0$.*
  - *For $n = 2$, 12 divides $2^4 - 2^2 = 12$.*
  - *For $n = 3$, 12 divides $3^4 - 3^2 = 72$.*
  - *For $n = 4$, 12 divides $4^4 - 4^2 = 240$.*
  - *For $n = 5$, 12 divides $5^4 - 5^2 = 600$.*
  - *For $n = 6$, 12 divides $6^4 - 6^2 = 1260$.*

□

Cont.

- For $k \geq 6$, assume $12 | (m^4 - m^2)$ for $1 \leq m \leq k$. (i.e., $S_1, \ldots, S_k$ are true.)
  We must show S_{k+1} is true, that is $12 \big| \big((k+1)^4 - (k+1)^2\big)$. Now, S_{k-5} being true means $12 \big| \big((k-5)^4 - (k-5)^2\big)$. To simplify, put $k - 5 = \ell$ so $12 | (\ell^4 - \ell^2)$, meaning $\ell^4 - \ell^2 = 12a$ for $a \in \mathbb{Z}$, and $k + 1 = \ell + 6$.

□

Then,

$$\begin{aligned}(k+1)^4-(k+1)^2 &= (\ell+6)^4-(\ell+6)^2\\ &= \ell^4+24\ell^3+216\ell^2+864\ell+1296\\ &\quad -(\ell^2+12\ell+36)\\ &= (\ell^4-\ell^2)+24\ell^3+216\ell^2+852\ell+1260\\ &= 12a+24\ell^3+216\ell^2+852\ell+1260\\ &= 12(a+2\ell^3+18\ell^2+71\ell+105).\end{aligned}$$

Because $(a+2\ell^3+18\ell^2+71\ell+105)\in\mathbb{Z}$, we get $12\big|\big((k+1)^4-(k+1)^2\big)$.

Outline for Proof by Smallest Counterexample

Proposition The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

Proof. (Smallest counterexample)
(1) Check that the first statement S_1 is true.
(2) For the sake of contradiction, suppose not every S_n is true.
(3) Let $k > 1$ be the smallest integer for which S_k is **false**.
(4) Then S_{k-1} is true and S_k is false. Use this to get a contradiction. ■

Proposition If $n \in \mathbb{N}$, then $4 \mid (5^n - 1)$.

Proof. We use proof by smallest counterexample. (We will number the steps to match the outline, but that is not usually done in practice.)

(1) If $n = 1$, then the statement is $4 \mid (5^1 - 1)$, or $4 \mid 4$, which is true.

(2) For sake of contradiction, suppose it's not true that $4 \mid (5^n - 1)$ for all n.

(3) Let $k > 1$ be the smallest integer for which $4 \nmid (5^k - 1)$.

(4) Then $4 \mid (5^{k-1} - 1)$, so there is an integer a for which $5^{k-1} - 1 = 4a$. Then:

$$\begin{aligned} 5^{k-1} - 1 &= 4a \\ 5(5^{k-1} - 1) &= 5 \cdot 4a \\ 5^k - 5 &= 20a \\ 5^k - 1 &= 20a + 4 \\ 5^k - 1 &= 4(5a + 1) \end{aligned}$$

This means $4 \mid (5^k - 1)$, a contradiction, because $4 \nmid (5^k - 1)$ in Step 3. Thus, we were wrong in Step 2 to assume that it is untrue that $4 \mid (5^n - 1)$ for every n. Therefore $4 \mid (5^n - 1)$ is true for every n. ■

Try to prove the following statement using strong induction:

Proposition

If a tree has n vertices, then it has $n - 1$ edges.

Proofs Involving Sets

1. For every integer $n \in \mathbb{N}$, it follows that $1+2+3+4+\cdots+n = \frac{n^2+n}{2}$.

2. For every integer $n \in \mathbb{N}$, it follows that $1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

3. For every integer $n \in \mathbb{N}$, it follows that $1^3+2^3+3^3+4^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$.

4. If $n \in \mathbb{N}$, then $1\cdot2+2\cdot3+3\cdot4+4\cdot5+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3}$.

5. If $n \in \mathbb{N}$, then $2^1+2^2+2^3+\cdots+2^n = 2^{n+1}-2$.

6. For every natural number n, it follows that $\sum_{i=1}^{n}(8i-5) = 4n^2-n$.

7. If $n \in \mathbb{N}$, then $1\cdot3+2\cdot4+3\cdot5+4\cdot6+\cdots+n(n+2) = \frac{n(n+1)(2n+7)}{6}$.

8. If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

9. For any integer $n \geq 0$, it follows that $24 \mid (5^{2n} - 1)$.

10. For any integer $n \geq 0$, it follows that $3 \mid (5^{2n} - 1)$.

11. For any integer $n \geq 0$, it follows that $3 \mid (n^3 + 5n + 6)$.

12. For any integer $n \geq 0$, it follows that $9 \mid (4^{3n} + 8)$.

Proofs Involving Sets

13. For any integer $n \geq 0$, it follows that $6 \mid (n^3 - n)$.

14. Suppose $a \in \mathbb{Z}$. Prove that $5 \mid 2^n a$ implies $5 \mid a$ for any $n \in \mathbb{N}$.

15. If $n \in \mathbb{N}$, then $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$.

16. For every natural number n, it follows that $2^n + 1 \leq 3^n$.

17. Suppose $A_1, A_2, \ldots A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}$.

18. Suppose $A_1, A_2, \ldots A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

19. Prove that $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

20. Prove that $(1+2+3+\cdots+n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$ for every $n \in \mathbb{N}$.