Abstract Mathematics
Lecture 15

Oktay Ölmez

Mathematical Induction

Proof by Smallest Counterexample

It is a hybrid of induction and proof by contradiction.

Outline for Proof by Smallest Counterexample

Proposition

The statements $S_1, S_2, S_3, S_4, \ldots$ nermelerinin hepsi dorudur.

Proof.

(Smallest counterexample)).
(1) Check that the first statement S_1 is true.
(2) For the sake of contradiction, suppose not every S_n is true.
(3) Let $k > 1$ be the smallest integer for which S_k is **false**.
(4) Then S_{k-1} is true and S_k is false. Use this to get a contradiction. □

Example

Proposition

If $n \in \mathbb{N}$, then $4|(5^n - 1)$.

Proof.

We use proof by smallest counterexample.

- *If $n = 1$, then the statement is $4|(5^1 - 1)$, or $4|4$, which is true.*
- *For sake of contradiction, suppose its not true that $4|(5^n - 1)$ for all n.*

□

Mathematical Induction

Cont.

- Let $k > 1$ be the smallest integer for which $4 \nmid (5^k - 1)$.
- Then $4|(5^{k-1} - 1)$, so there is an integer a for which $5^{k-1} - 1 = 4a$.

$$
\begin{aligned}
5^{k-1} - 1 &= 4a \\
5(5^{k-1} - 1) &= 5 \cdot 4a \\
5^k - 5 &= 20a \\
5^k - 1 &= 20a + 4 \\
5^k - 1 &= 4(5a + 1)
\end{aligned}
$$

This means $4|(5^k - 1)$, a contradiction, because $4 \nmid (5^k - 1)$ in Step 3. Thus, we were wrong in Step 2 to assume that it is untrue that $4|(5^k - 1)$ for every n. Therefore $4|(5^k - 1)$ is true for every n.

□

The Fundamental Theorem of Arithmetic

The fundamental theorem of arithmetic states that any integer greater than 1 has a unique prime factorization.

For example, 12 factors into primes as $12 = 2 \cdot 2 \cdot 3$, and moreover any factorization of 12 into primes uses exactly the primes 2, 2 and 3.

Theorem

(Fundamental Theorem of Arithmetic) Any integer $n > 1$ *has a unique prime factorization. Unique means that if* $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ *and* $n = a_1 \cdot a_2 \cdot a_3 \cdots a_l$*' are two prime factorizations of* n*, then* $k = l$*, and the primes* a_i *and* p_i *are the same, except that they may be in different orders.*

Proof.

The proof combines the techniques of induction, cases, minimum counterexample and the idea of uniqueness of existence. □

see for the full proof at page 192 of "Book of Proof".

Fibonacci Numbers

Leonardo Pisano, now known as Fibonacci, was a mathematician born around 1175 in what is now Italy.

he is best known today for a number sequence that he described as

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377, \ldots$$

and known as **Fibonacci sequence**

We denote the nth term of this sequence as F_n. Thus $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_7 = 13$ and so on.

Notice that the Fibonacci sequence is entirely determined by the rules $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

We introduce Fibonaccis sequence here partly because it is somethingeveryone should knowabout, but also because it is a great source of induction problems.

Proposition

The Fibonacci sequence obeys $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$

Proof.

We will prove this with mathematical induction.

- If $n = 1$ we have $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = F_2^2 - F_2F_1 - F_1^2 = 1^2 - 1 \cdot 1 - 1^2 = -1 = (-1)^1 = (-1)^n$, so indeed $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ is true when $n = 1$.

□

Cont.

- Let $k \in \mathbb{N}$. Observe that

$$\begin{aligned} F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2 &= (F_{k+1} + F_k)^2 - (F_{k+1} + F_k)F_{k+1} \\ &\quad - F_{k+1}^2 \\ &= F_{k+1}^2 + 2F_{k+1}F_k + F_k^2 - F_{k+1}^2 \\ &\quad - F_kF_{k+1} - F_{k+1}^2 \\ &= -F_{k+1}^2 + F_{k+1}F_k + F_k^2 \\ &= -(F_{k+1}^2 - F_{k+1}F_k - F_k^2) \\ &= -(-1)^k \\ &= (-1)(-1)^k \\ &= (-1)^{k+1} \end{aligned}$$

Therefore $F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2 = (-1)^{k+1}$.

It follows by induction that $n \in \mathbb{N}$ iin $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ for every $n \in \mathbb{N}$

Prove the following statements with either induction, strong induction or proof by smallest counterexample.

- Prove that $n \in \mathbb{N}$ iin $1 + 2 + 3 + 4 + \cdots + n = \frac{n^2+n}{2}$ for every positive integer n.
- Prove that $n \in \mathbb{N}$ iin $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer n.
- Prove that $n \in \mathbb{N}$ iin $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.
- If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.
- If $n \in \mathbb{N}$, then $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.