Abstract Mathematics
Lecture 16

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Equivalence Relations:

Definition

A relation R on a set A is an equivalence relation if it is reflexive, symmetric and transitive.

Definition

Suppose R is an equivalence relation on a set A. Given any element $a \in A$, the equivalence class containing a is the subset $\{x \in A : xRa\}$ of A consisting of all the elements of A that relate to a. This set is denoted as $[a]$. Thus the equivalence class containing a is the set $[a] = \{x \in A : xRa\}$.

Example

Let $A = \{-1, 1, 2, 3, 4\}$ *and* R_1, R_2, R_3 *ve* R_4 *described as below*

$R_2 =$
$\{(-1,-1),(1,1),(2,2),(3,3),(4,4),(-1,1),(1,-1),(-1,3),(3,-1),(1,3)$
$,(3,1),(2,4),(4,2)\}$
$R_1 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4)\}$

Example

$$R_4 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),(1,3),(3,1),(2,4),(4,2)\}$$

$R_3 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),(3,4),(4,3),(2,3),(3,2),(2,4),(4,2)\}$

Relations

Example

Consider the relation R_1. The equivalence class containing 2 is the set $[2] = \{x \in A : xR_1 2\}$. Other equivalence classes for R_1 are $[-1] = \{-1\}$, $[1] = \{1\}$, $[3] = \{3\}$ and $[4] = \{4\}$. Thus this relation has five separate equivalence classes.

Example

The relation R_4 has three equivalence classes. They are $[-1] = \{-1\}$, $[1] = [3] = \{1, 3\}$ and $[2] = [4] = \{2, 4\}$.

Example

Let P be the set of all polynomials with real coefficients. Define a relation R on P as follows. Given $f(x), g(x) \in P$, let $f(x)Rg(x)$ mean that $f(x)$ and $g(x)$ have the same degree. We can write
$[3x^2 + 2] = \left\{ax^2 + bx + c : a, b, c \in \mathbb{R}, a \neq 0\right\}$.

Equivalence Classes and Partitions:

Theorem

Suppose R is an equivalence relation on a set A. Suppose also that $a, b \in A$. Then $[a] = [b]$ if and only if aRb.

Example

To illustrate the Theorem, consider the equivalence classes of $\pmod 3$.

$$[-3] = [9] = \{\ldots, -3, 0, 3, 6, 9, \ldots\}.$$

An equivalence relation on a set A divides A into various equivalence classes. There is a special word for this kind of situation.

Definition

A partition of a set A is a set of non-empty subsets of A, such that the union of all the subsets equals A, and the intersection of any two different subsets is $\emptyset$.

Example

Let $A = \{a, b, c, d\}$. *One partition of* A *is* $\{\{a, b\}, \{c\}, \{d\}\}$. *This is a set of three subsets* $\{a, b\}$, $\{c\}$ *ve* $\{d\}$ *of* A. *The union of the three subsets equals* A*; the intersection of any two subsets is* $\emptyset$.

Relations

Theorem

Suppose R is an equivalence relation on a set A. Then the set $\{[a] : a \in A\}$ of equivalence classes of R forms a partition of A.

The theorem says the equivalence classes of any equivalence relation on a set A form a partition of A. Conversely, any partition of A describes an equivalence relation R where xRy if and only if x and y belong to the same set in the partition.

Consider the equivalence classes of the relation $\equiv (mod 5)$. There are five equivalence classes, as follows:

$$\begin{aligned}
[0] &= \{x \in \mathbb{Z} : x \equiv 0 mod 5\} = \{\ldots, -10, -5, 0, 5, 10, 15, \ldots\}, \\
[1] &= \{x \in \mathbb{Z} : x \equiv 1 mod 5\} = \{\ldots, -9, -4, 0, 6, 11, 16, \ldots\}, \\
[2] &= \{x \in \mathbb{Z} : x \equiv 2 mod 5\} = \{\ldots, -8, -3, 0, 7, 12, 17, \ldots\}, \\
[3] &= \{x \in \mathbb{Z} : x \equiv 3 mod 5\} = \{\ldots, -7, -2, 0, 8, 13, 18, \ldots\}, \\
[4] &= \{x \in \mathbb{Z} : x \equiv 4 mod 5\} = \{\ldots, -6, -1, 0, 9, 14, 19, \ldots\}.
\end{aligned}$$

These five classes form a set, which we shall denote as $\mathbb{Z}_5$. Thus

$$\mathbb{Z}_5 = \{ [0], [1], [2], [3], [4] \}$$

Definition

Let $n \in \mathbb{N}$. The equivalence classes of the equivalence relation $\equiv \text{mod} n$ are $[0], [1], [2], \ldots, [n-1]$. The integers modulo n is the set $\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$. Elements of $\mathbb{Z}_n$ can be added by the rule $[a] + [b] = [a+b]$ and multiplied by the rule $[a] \cdot [b] = [ab]$.

Relations Between Sets:

Definition

A relation from a set A to a set B is a subset $R \subseteq A \times B$. We often abbreviate the statement $(x, y) \in R$ as xRy. The statement $(x, y) \notin R$ is abbreviated as $x \not R y$.

Example

Suppose $A = \{1, 2\}$ and $B = \mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Then $R = \Big\{(1, \{1\}), (2, \{2\}), (1, \{1, 2\}), (2, \{1, 2\})\Big\} \subseteq A \times B$ is a relation from A to B.