Abstract Mathematics
Lecture 16

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Relations

In mathematics there are endless ways that two entities can be related to each other.

$$
\begin{array}{cccccc}
5 < 10 & 5 \leq 5 & 6 = \frac{30}{5} & 5|80 & 7 > 4 & x \neq y \\
a \equiv b \text{ Mod} n & 6 \in \mathbb{Z} & X \subseteq Y & \pi \approx 3.14 & 0 \geq -1 & \sqrt{2} \notin \mathbb{Z}
\end{array}
$$

Symbols such as $<, \leq, =, |, \nmid, \geq, >, \in, \subset$, etc., are called relations because they convey relationships among things.

Relations

Definition

A relation on a set A *is a subset* $R \subseteq A \times A$*. We often abbreviate the statement* $(x, y) \in R$ *as* xRy*. The statement* $(x, y) \notin R$ *is abbreviated as* $x \not R y$*.*

Relations

Example

Let $A = \{1, 2, 3, 4\}$*, and consider the following set:*

$R =$
$\{\,(1,1),\ (2,1),\ (2,2),\ (3,3),\ (3,2),\ (3,1),\ (4,4),\ (4,3),\ (4,2),\ (4,1)\,\} \subseteq$
$A \times A.$

The set R is a relation on A, by Definition.

Example

Let $B = \{0, 1, 2, 3, 4, 5\}$, *and consider the following set:*

$$U = \{(1,3),\ (3,3),\ (5,2),\ (2,5),\ (4,2)\} \subseteq B \times B.$$

Then U is a relation on B because $U \subseteq B \times B$.

Relations

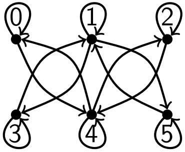
Example

Consider the set $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - y \in \mathbb{N}\} \subseteq \mathbb{Z} \times \mathbb{Z}$. *This is the* $>$ *relation on the set* $A = \mathbb{Z}$. *It is infinite because there are infinitely many ways to have* $x > y$ *where* x *and* y *are integers.*

The statement $(x, y) \in U$ is then represented by an arrow pointing from x to y, a graphic symbol meaning x relates to y.

Relations

Here is a diagram for a relation R on a set A. Write the sets A and R.



Properties of Relations

Definition

Suppose R is a relation on a set A.

- *Relation R is reflexive if xRx for every $x \in A$.*
  That is, R is reflexive if $\forall x \in A, xRx$.
- *Relation R is symmetric if xRy implies yRx for all $x, y \in A$.*
  That is, R is symmetric if $\forall x, y \in A, xRy \rightarrow yRx$.
- *Relation R is transitive if whenever xRy and yRz, then also xRz.*
  That is, R is transitive if $\forall x, y, z \in A, \big((xRy) \wedge (yRz)\big) \rightarrow xRz$.

Relations

To illustrate this, lets consider the set $A = \mathbb{Z}$ and the relations $<$, $\leq$, $=$, $|$, $\nmid$ and $\neq$.

We have the following table.

Relations on $\mathbb{Z}$:	$<$	$\leq$	$=$	$\mid$	$\nmid$	$\neq$
Reflexive	no	yes	yes	yes	no	no
Symmetric	no	no	yes	no	no	yes
Transitive	yes	yes	yes	yes	no	no

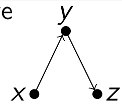
How to spot the various properties of a relation from its diagram.

1. A relation is reflexive if for each point x . . . • x . . . there is a loop at x: 

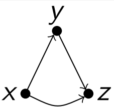
2. A relation is symmetric if whenever there is an arrow from x to y. . . 

. . . there is also an arrow from y back to x: 

3. A relation is transitive if whenever there are arrows from x to y and y to z...



...there is also an arrow from x to z:



(If $x = z$, this means that if there are arrows from x to y and from y to x...

...there is also a loop from x back to x.)