Abstract Mathematics
Lecture 17

Oktay Ölmez

Functions

Lets start on familiar ground. Consider the function $f(x) = x^2$ from $\mathbb{R} to \mathbb{R}$.

Its graph is the set of points $R = \{(x, x^2) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$.

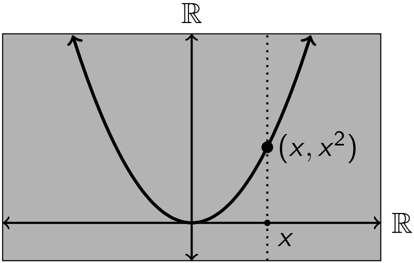


Figure: A familiar function

Functions

This example illustrates three things.

First, a function can be viewed as sending elements from one set A to another set B.

Second, such a function can be regarded as a relation from A to B.

Third, for every input value n, there is exactly one output value $f(n)$.

Functions

Definition

Suppose A and B are sets. A function f from A to B (denoted as $f : A \to B$) is a relation $f \subseteq A \times B$ from A to B, satisfying the property that for each $a \in A$ the relation f contains exactly one ordered pair of form (a, b). The statement $(a, b) \in f$ is abbreviated $f(a) = b$.

Example

The function $f : \mathbb{Z} \to \mathbb{N}$, where $f(n) = |n| + 2$

This is a relation from $\mathbb{Z} to \mathbb{N}$, and indeed given any $a \in \mathbb{Z}$ the set f contains exactly one ordered pair $(a, |a| + 2)$ whose first coordinate is a.

Functions

Definition

For a function $f : A \to B$, the set A is called the domain of f. The set B is called the codomain of f. The range of f is the set $\{f(a) : a \in A\} = \{b : (a, b) \in f\}$.

Example

Let $A = \{p, q, r, s\}$ and $B = \{0, 1, 2\}$, and

$$f = \{(p, 0), (q, 1), (r, 2), (s, 2)\} \subseteq A \times B.$$

This is a function $f : A \to B$ because each element of A occurs exactly once as a first coordinate of an ordered pair in f.

The domain of this function is $\{p, q, r, s\}$.

The codomain and range are both $\{0, 1, 2\}$.

Functions

Example

Say a function $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ is defined as $\varphi(m, n) = 6m - 9n$. What is the range of φ?

Definition

Two functions $f : A \to B$ and $g : A \to D$ are equal if $f = g$ (as sets). Equivalently, f g if and only if $f(x) = g(x)$ for every $x \in A$.

Observe that f and g can have different codomains and still be equal.

Consider the functions $f : \mathbb{Z} \to \mathbb{N}$ and $g : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = |x| + 2$ and $g(x) = |x| + 2$.

Functions

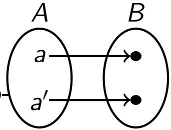
Definition

A function $f : A \to B$ is:

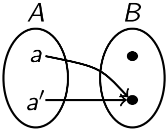
- **injective** *(or one-to-one) if for all* $a, a' \in A,\ a \neq a'$ *implies* $f(a) \neq f(a')$*;*
- **surjective** *(or onto B) if for every* $b \in B$ *there is an* $a \in A$ *with* $f(a) = b$*;*
- **bijective** *if f is both injective and surjective.*

Functions

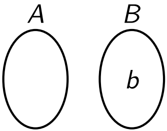
Injective means that for any two $a, a' \in A$, this happens ...



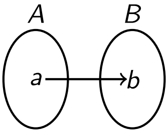
...and not this:



Surjective means that for any $b \in B$...



...this happens:



Example

The function $f(x) = x^2$ is not injective because $-2 \neq 2$, but $f(-2) = f(2)$.

Nor is it surjective, for if $b = -1$, then there is no $a \in \mathbb{R}$ with $f(a) = b$.

How to show a function $f : A \to B$ is injective:

Direct approach: Suppose $a, a' \in A$ and $a \neq a'$. $\vdots$ Therefore $f(a) \neq f(a')$.	**Contrapositive approach:** Suppose $a, a' \in A$ and $f(a) = f(a')$. $\vdots$ Therefore $a = a'$.

How to show a function $f : A \to B$ is surjective:

Suppose $b \in B$.
[Prove there exists $a \in A$ for which $f(a) = b$.]

Example

Show that the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x} + 1$ is injective but not surjective.

Example

Show that the function $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{1\}$ defined as $f(x) = \frac{1}{x} + 1$ is injective and surjective (hence bijective).

Functions

The Pigeonhole Principle Revisited:

The pigeonhole principle is motivated by a simple thought experiment: Imagine there is a set A of pigeons and a set B of pigeonholes, and all the pigeons fly into the pigeonholes. You can think of this as describing a function $f : A \to B$, where pigeon p flies into pigeonhole $f(p)$.

Definition

The Pigeonhole Principle *(function version)*
Suppose A and B are finite sets and $f : A \to B$ is any function.

- *If $|A| > |B|$, then f is not injective.*
- *If $|A| < |B|$, then f is not surjective.*

Example

Though the pigeonhole principle is obvious, it can be used to prove some things that are not so obvious.

Proposition

There are at least two Texans with the same number of hairs on their heads.

Proof.

We will use two facts. First, the population of Texas is more than twenty million. Second, it is a biological fact that every human head has fewer than one million hairs.

Let A be the set of all Texans, and let $B = \{0, 1, 2, 3, 4, \ldots, 1000000\}$. Let $f : A \to B$ be the function for which $f(x)$ equals the number of hairs on the head of x.

Cont.

Since $|A| > |B|$, the pigeonhole principle asserts that f is not injective. Thus there are two Texans x and y for whom $f(x) = f(y)$, meaning that they have the same number of hairs on their heads. □

Composition

Definition

Suppose $f : A \to B$ and $g : B \to C$ are functions with the property that the codomain of f equals the domain of g. The composition of f with g is another function, denoted as $g \circ f$ and defined as follows:
If $x \in A$, then $g \circ f(x) = g(f(x))$. Therefore $g \circ f$ sends elements of A to elements of C, so $g \circ f : A \to C$.

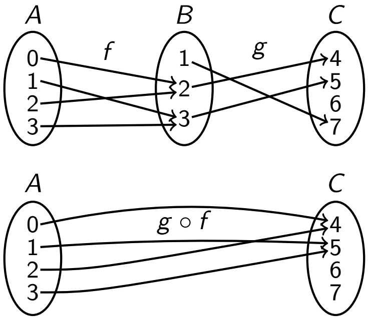


Figure: Composition of two functions

Functions

Example

Suppose $A = \{a, b, c\}$, $B = \{0, 1\}$ *and* $C = \{1, 2, 3\}$. *Let* $f : A \to B$ *be the function* $f = \{(a, 0), (b, 1), (c, 0)\}$, *and let* $g : B \to C$ *be* $g = \{(0, 3), (1, 1)\}$. *Then* $g \circ f = \{(a, 3), (b, 1), (c, 3)\}$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ *be defined as* $f(x) = x^2 + x$, *and* $g : \mathbb{R} \to \mathbb{R}$ *be defined as* $g(x) = x + 1$. *Then* $g \circ f : \mathbb{R} \to \mathbb{R}$ *is the function defined by the formula* $g \circ f(x) = g(f(x)) = g(x^2 + x) = x^2 + x + 1$.

Functions

These theorems are several facts about composition that you are likely to encounter in your future study of mathematics.

Theorem

Composition of functions is associative. That is if $f : A \to B$, $g : B \to C$ *and* $h : C \to D$, *then* $(h \circ g) \circ f = h \circ (g \circ f)$.

Theorem

Suppose $f : A \to B$ *and* $g : B \to C$. *If both* f *and* g *are injective, then* $g \circ f$ *is injective. If both* f *and* g *are surjective, then* $g \circ f$ *is surjective.*

Functions

1. Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (3m - 4n, 2m + n)$ and $g(m, n) = (5m + n, m)$. Find the formulas for $g \circ f$ and $f \circ g$.
2. Consider the functions $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f(m, n) = m + n$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $g(m) = (m, m)$. Find the formulas for $g \circ f$ and $f \circ g$.
3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = (xy, x^3)$. Find a formula for $f \circ f$.