

# Abstract Mathematics

## Lecture 18

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## Inverse Functions:

### Definition

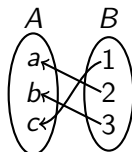
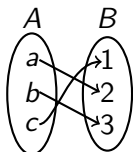
For a set  $A$ , the identity function on  $A$  is the function  $i_A : A \rightarrow A$  defined as  $i_A(x) = x$  for every  $x \in A$ .

### Example

If  $A = \{1, 2, 3\}$ , then  $i_A = \{(1, 1), (2, 2), (3, 3)\}$ . Also  $i_{\mathbb{Z}} = \{(n, n) : n \in \mathbb{Z}\}$

## Definition

Given a relation  $R$  from  $A$  to  $B$ , the **inverse relation** of  $R$  is the relation from  $B$  to  $A$  defined as  $R^{-1} = \{(y, x) : (x, y) \in R\}$ . In other words, the inverse of  $R$  is the relation  $R^{-1}$  obtained by interchanging the elements in every ordered pair in  $R$ .



$$f = \{(a, 2), (b, 3), (c, 1)\} \quad f^{-1} = \{(2, a), (3, b), (1, c)\}$$

If  $g$  is a function, then it must be bijective in order for its inverse relation  $g^{-1}$  to be a function. Conversely, if a function is bijective, then its inverse relation is easily seen to be a function.

## Theorem

*Let  $f : A \rightarrow B$  be a function. Then  $f$  is bijective if and only if the inverse relation  $f^{-1}$  is a function from  $B$  to  $A$ .*

This leads to the following definition.

## Definition

*If  $f : A \rightarrow B$  is bijective then its inverse is the function  $f^{-1} : B \rightarrow A$ . The functions  $f$  and  $f^{-1}$  obey the equations  $f^{-1} \circ f = i_A$  and  $f \circ f^{-1} = i_B$ .*

## Example

The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f(x) = x^3 + 1$  is bijective. Find its inverse.

We begin by writing  $y = x^3 + 1$ . Now interchange variables to obtain  $x = y^3 + 1$ . Solving for  $y$  produces  $y = \sqrt[3]{x-1}$ . Thus

$$f^{-1}(x) = \sqrt[3]{x-1}$$

. (You can check your answer by computing

$$f^{-1}(f(x)) = \sqrt[3]{f(x)-1} = \sqrt[3]{x^3+1-1} = x.$$

Therefore  $f^{-1}(f(x)) = x$ . Any answer other than  $x$  indicates a mistake.)

## Image and Preimage

### Definition

Suppose  $f : A \rightarrow B$  is a function.

- If  $X \subseteq A$ , the image of  $X$  is the set  $f(X) = \{f(x) : x \in X\} \subseteq B$ .
- If  $Y \subseteq B$ , the preimage of  $Y$  is the set  $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$ .

## Example

Let  $f : \{s, t, u, v, w, x, y, z\} \rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  be  
 $f = \{(s, 4), (t, 8), (u, 8), (v, 1), (w, 2), (x, 4), (y, 6), (z, 4)\}$ .

*This  $f$  is neither injective nor surjective, so it certainly is not invertible. Be sure you understand the following statements.*

①  $f(\{s, t, u, z\}) = \{8, 4\}$

②  $f(\{s, x, z\}) = \{4\}$

③  $f(\{s, v, w, y\}) = \{1, 2, 4, 6\}$

④  $f(\emptyset) = \emptyset$

⑤  $f^{-1}(\{4\}) = \{s, x, z\}$

⑥  $f^{-1}(\{4, 9\}) = \{s, x, z\}$

⑦  $f^{-1}(\{9\}) = \emptyset$

⑧  $f^{-1}(\{1, 4, 8\}) = \{s, t, u, v, x, z\}$



You will likely encounter the following results. For now, you are asked to prove them.

## Theorem

Given  $f : A \rightarrow B$ , let  $W, X \subseteq A$ , and  $Y, Z \subseteq B$ . Then

$$\textcircled{1} \quad f(W \cap X) \subseteq f(W) \cap f(X)$$

$$\textcircled{2} \quad f(W \cup X) = f(W) \cup f(X)$$

$$\textcircled{3} \quad X \subseteq f^{-1}(f(X))$$

$$\textcircled{4} \quad f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$$

$$\textcircled{5} \quad f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$$

$$\textcircled{6} \quad f(f^{-1}(Y)) \subseteq Y.$$