Abstract Mathematics
Lecture 18

Oktay Ölmez

Inverse Functions:

Definition

For a set A, the identity function on A is the function $i_A : A \to A$ defined as $i_A(x) = x$ for every $x \in A$.

Example

If $A = \{1, 2, 3\}$, then $i_A = \{(1, 1), (2, 2), (3, 3)\}$. Also $i_{\mathbb{Z}} = \{(n, n) : n \in \mathbb{Z}\}$

Functions

Definition

Given a relation R from A to B, the **inverse relation** *of R is the relation from B to A defined as $R^{-1} = \{(y, x) : (x, y) \in R\}$. In other words, the inverse of R is the relation R^{-1} obtained by interchanging the elements in every ordered pair in R.*

$f = \{(a, 2), (b, 3), (c, 1)\}$ $f^{-1} = \{(2, a), (3, b), (1, c)\}$

If g is a function, then it must be bijective in order for its inverse relation g^{-1} to be a function. Conversely, if a function is bijective, then its inverse relation is easily seen to be a function.

Theorem

Let $f : A \to B$ be a function. Then f is bijective if and only if the inverse relation f^{-1} is a function from B to A.

This leads to the following definition.

Definition

If $f : A \to B$ is bijective then its inverse is the function $f^{-1} : B \to A$. The functions f and f^{-1} obey the equations $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Functions

Example

The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^3 + 1$ is bijective. Find its inverse.

We begin by writing $y = x^3 + 1$. Now interchange variables to obtain $x = y^3 + 1$. Solving for y produces $y = \sqrt[3]{x-1}$. Thus

$$f^{-1}(x) = \sqrt[3]{x-1}$$

. (You can check your answer by computing

$$f^{-1}(f(x)) = \sqrt[3]{f(x)-1} = \sqrt[3]{x^3+1-1} = x.$$

Therefore $f^{-1}(f(x)) = x$. Any answer other than x indicates a mistake.)

Image and Preimage

Definition

Suppose $f : A \to B$ *is a function.*

- *If* $X \subseteq A$, *the image of* X *is the set* $f(X) = \{f(x) : x \in A\} \subseteq B$.
- *If* $Y \subseteq B$, *the preimage of* Y *is the set* $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.

Example

Let $f : \{s,t,u,v,w,x,y,z\} \to \{0,1,2,3,4,5,6,7,8,9\}$ *be* $f = \{(s,4),(t,8),(u,8),(v,1),(w,2),(x,4),(y,6),(z,4)\}$.
This f *is neither injective nor surjective, so it certainly is not invertible. Be sure you understand the following statements.*

1. $f(\{s,t,u,z\}) = \{8,4\}$
2. $f(\{s,x,z\}) = \{4\}$
3. $f(\{s,v,w,y\}) = \{1,2,4,6\}$
4. $f(\emptyset) = \emptyset$
5. $f^{-1}(\{4\}) = \{s,x,z\}$
6. $f^{-1}(\{4,9\}) = \{s,x,z\}$
7. $f^{-1}(\{9\}) = \emptyset$
8. $f^{-1}(\{1,4,8\}) = \{s,t,u,v,x,z\}$

Functions

You will likely encounter the following results. For now, you are asked to prove them.

Theorem

Given $f : A \to B$, *let* $W, X \subseteq A$, *and* $Y, Z \subseteq B$. *Then*

1. $f(W \cap X) \subseteq f(W) \cap f(X)$
2. $f(W \cup X) = f(W) \cup f(X)$
3. $X \subseteq f^{-1}\big(f(X)\big)$
4. $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$
5. $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$
6. $f\big(f^{-1}(Y)\big) \subseteq Y$.