Abstract Mathematics
Lecture 19

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The Triangle Inequality

Definitions in calculus and analysis use absolute value extensively.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Fundamental properties of absolute value include $|xy| = |x| \cdot |y|$ and $x \leq |x|$.
Another property—used often in proofs—is the triangle inequality

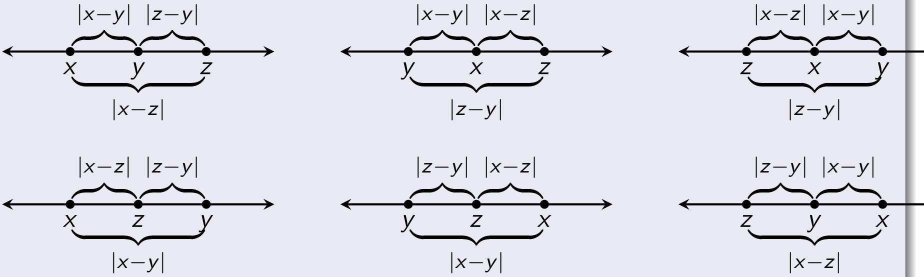
Definition

If $x, y, z \in \mathbb{R}$, *then* $|x - y| \leq |x - z| + |z - y|$.

Proofs in Calculus

Sketch Proof.

Observe in the diagrams below that regardless of the order of x, y, z on the number line, the inequality $|x-y| \leq |x-z| + |z-y|$ holds.



□

Proofs in Calculus

Definition of a Limit

We need to know how a certain function $f(x)$ behaves when x is close to some number c.

Limits are designed to deal with this type of problem.

Definition (Informal definition of a limit)

Suppose f is a function and c is a number. Then $\lim_{x \to c} f(x) = L$ means that $f(x)$ is arbitrarily close to L provided that x is sufficiently close to c.

Definition (Precise definition of a limit)

Suppose $f : X \to \mathbb{R}$ is a function, where $X \subseteq \mathbb{R}$, and $c \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = L$ means that for any real $\varepsilon > 0$ (no matter how small), there is a real number $\delta > 0$ for which $|f(x) - L| < \varepsilon$ provided that $0 < |x - c| < \delta$.

Proofs in Calculus

Example

Prove that $\lim_{x \to 2} (3x + 4) = 10$.

Proof.

Suppose $\varepsilon > 0$. *Note that* $|(3x + 4) - 10| = |3x - 6| = |3(x - 2)| = 3|x - 2|$. *So if* $\delta = \frac{\varepsilon}{3}$, *then* $0 < |x - 2| < \delta$ *yields* $|(3x + 4) - 10| = 3|x - 2| < 3\delta = 3\frac{\varepsilon}{3} = \varepsilon$. *In summary, for any* $\varepsilon > 0$, *there is* δ *for which* $0 < |x - 2| < \delta$ *implies* $|(3x + 4) - 10| < \varepsilon$. *By Definition,* $\lim_{x \to 2} (3x + 4) = 10$. □

Example

Prove that $\lim_{x \to 2} 5x^2 = 20$.

Limits That Do Not Exist Given a function f and a number c, there are two ways that $\lim_{x \to c} f(x) = L$ can be false.

First, there may be a different number $M \neq L$ for which $\lim_{x \to c} f(x) = M$.

Second, it may be that Statement

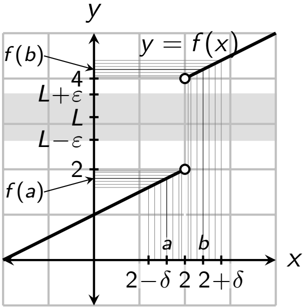
$$\forall \varepsilon > 0, \exists \delta > 0, \left(0 < |x - c| < \delta \right) \Rightarrow \left(|f(x) - L| < \varepsilon \right). \tag{1}$$

is false for all $L \in \mathbb{R}$. In such a case we say that $\lim_{x \to c} f(x)$ does not exist.

Example

Suppose $f(x) = \frac{x}{2} + \frac{|x-2|}{x-2} + 2$. *Prove* $\lim_{x \to 2} f(x)$ *does not exist.*

Notice that $f(2)$ is not defined, as it involves division by zero. Also, $f(x)$ behaves differently depending on whether x is to the right or left of 2.



Suppose for the sake of contradiction that $\lim_{x\to 2} f(x) = L$, where L is a real number. Let $\varepsilon = \frac{1}{2}$.

By Definition, there is a real number $\delta > 0$ for which $0 < |x-2| < \delta$ implies $\left|f(x) - L\right| < \frac{1}{2}$.

Put $a = 2 - \frac{\delta}{2}$, so $0 < |a-2| < \delta$. Hence $\left|f(a) - L\right| < \frac{1}{2}$.
Put $b = 2 + \frac{\delta}{2}$, so $0 < |b-2| < \delta$. Hence $\left|f(b) - L\right| < \frac{1}{2}$.

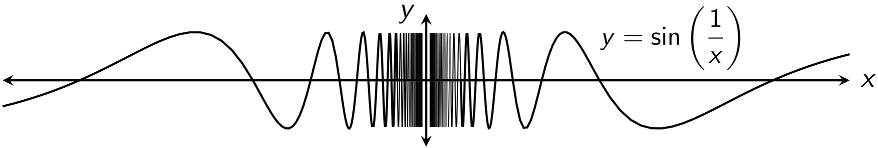
Further, $f(a) < 2$ and $f(b) > 4$, so $2 < |f(b) - f(a)|$. With this we get a contradiction $2 < 1$, as follows:

$2 < \left|f(b) - f(a)\right| = \left|\big(f(b) - L\big) - \big(f(a) - L\big)\right| \leq \left|f(b) - L\right| + \left|f(a) - L\right| < \frac{1}{2} + \frac{1}{2} = 1$

Proofs in Calculus

Example

Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ *does not exist.*



As x approaches 0, the number $\frac{1}{x}$ grows bigger, approaching infinity, so $\sin\left(\frac{1}{x}\right)$ just bounces up and down, faster and faster the closer x gets to 0.

Intuitively, we would guess that the limit does not exist, because $\sin\left(\frac{1}{x}\right)$ does not approach any single number as x approaches 0.

Limit Laws:

Our first limit law concerns the constant function $f(x) = a$ where $a \in \mathbb{R}$. Its graph is a horizontal line with y-intercept a.

Theorem (Constant function rule)

If $a \in \mathbb{R}$, then $\lim_{x \to c} a = a$.

The identity function $f(x) = x$.

Theorem (Identity Function Rule)

If $c \in \mathbb{R}$, then $\lim_{x \to c} x = c$.

Proofs in Calculus

Theorem (Constant multiple rule)

If $\lim_{x \to c} f(x)$ *exists, and* $a \in \mathbb{R}$*, then* $\lim_{x \to c} af(x) = a \lim_{x \to c} f(x)$*.*

Theorem (Sum rule)

If both $\lim_{x \to c} f(x)$ *and* $\lim_{x \to c} g(x)$ *exist, then*
$\lim_{x \to c} \big(f(x) + g(x)\big) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$*.*

Theorem (Difference rule)

If both $\lim_{x \to c} f(x)$ *and* $\lim_{x \to c} g(x)$ *exist, then*
$$\lim_{x \to c} \left(f(x) - g(x)\right) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x).$$

Theorem (Multiplication rule)

If both lim $\lim_{x \to c} f(x)$ *and* $\lim_{x \to c} g(x)$ *exist, then*
$$\lim_{x \to c} f(x)g(x) = \left(\lim_{x \to c} f(x)\right) \cdot \left(\lim_{x \to c} g(x)\right).$$

Theorem (Division Rule)

If both $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exist, and* $\lim_{x\to c} g(x) \neq 0$*, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Example

Find $\displaystyle\lim_{x\to 1} \frac{\frac{1}{x}-1}{1-x}$.

Here x approaches 1, but simply plugging in $x=1$ gives $\frac{\frac{1}{1}-1}{1-1}=\frac{0}{0}$ (undefined). So we apply whatever algebra is needed to cancel the denominator $1-x$, and follow this with limit laws:

$$\begin{aligned}
\lim_{x\to 1} \frac{\frac{1}{x}-1}{1-x} &= \lim_{x\to 1} \frac{\frac{1}{x}-1}{1-x}\,\frac{x}{x} && \left(\text{multiply quotient by } 1=\tfrac{x}{x}\right)\\
&= \lim_{x\to 1} \frac{(1-x)}{(1-x)x} && (\text{distribute } x \text{ on top})\\
&= \lim_{x\to 1} \frac{1}{x} && (\text{cancel the } (1-x))\\
&= \frac{\lim_{x\to 1} 1}{\lim_{x\to 1} x} = \frac{1}{1} = 1. && (\text{apply limit laws})
\end{aligned}$$