Abstract Mathematics
Lecture 21

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Cardinality of Sets

This lecture is all about cardinality of sets. At first this looks like a very simple concept. To find the cardinality of a set, just count its elements.

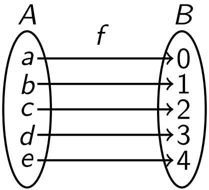
If $A = \{a, b, c, d\}$ then $|A| = 4$ and $B = \{n \in \mathbb{Z} : -5 \leq n \leq 5\}$ then $|B| = 11$.

Actually, the idea of cardinality becomes quite subtle when the sets are infinite.

Sets with Equal Cardinalities:

Definition

Two sets A and B have the same cardinality, written $|A| = |B|$, if there exists a bijective function $f : A \to B$. If no such bijective f exists, then the sets have unequal cardinalities, written $|A| \neq |B|$.



Cardinality of Sets

Example

The sets $A = \{n \in \mathbb{Z} : 0 \leq n \leq 5\}$ *and* $B = \{n \in \mathbb{Z} : -5 \leq n \leq 0\}$ *have the same cardinality because there is a bijective function* $f :$ $f : A \to B$ *given by the rule* $f(n) = -n$.

We emphasize and reiterate that definition applies to finite as well as infinite sets.

Example

$|\mathbb{N}| = |\mathbb{Z}|$. *To see why this is true, notice that the following table describes a bijection* $f : \mathbb{N} \to \mathbb{Z}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(n)$	0	1	-1	2	-2	3	-3	4	-4	5	-5	6	-6	7	-7

Cardinality of Sets

Try to explain the following theorems...

Theorem

There exists a bijection $f : \mathbb{N} \to \mathbb{Z}$*. Therefore* $|\mathbb{N}| = |\mathbb{Z}|$*.*

Theorem

There exists no bijection $f : \mathbb{N} \to \mathbb{R}$*. Therefore* $|\mathbb{N}| \neq |\mathbb{R}|$*.*

This is our first indication of how there are different kinds of infinities.

Both $\mathbb{N}$ and $\mathbb{R}$ are infinite sets, yet $|\mathbb{N}| \neq |\mathbb{R}|$.

The next example shows that the intervals $(0, 1)$ and $(0, \infty)$ on $\mathbb{R}$ have the same cardinality.

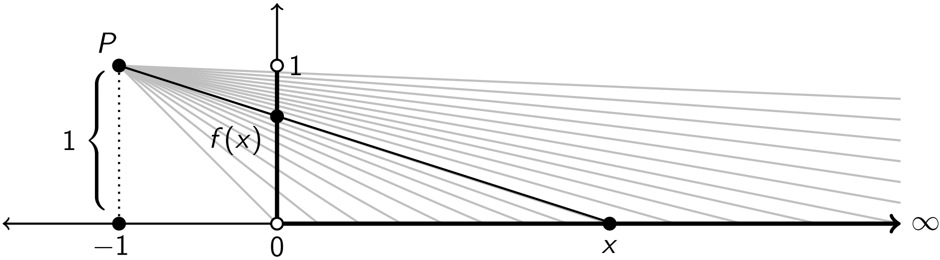


Figure: A bijection $f : (0, \infty) \to (0, 1)$. Imagine a light source at point P. Then $f(x)$ is the point on the y-axis whose shadow is x.

Example

Show that $|(0,1)| = |(0,1)|$.

To accomplish this, we need to show that there is a bijection $f : (0,\infty) \to (0,1)$. We can take

$$f(x) = \frac{x}{x+1}$$

You have to show that $f(x)$ is a bijective.

Cardinality of Sets

Countable and Uncountable Sets:

Let's summarize what we know...

- $|A| = |B|$ if and only if there exists a bijection $f : A \to B$.
- $|\mathbb{N}| = |\mathbb{Z}|$ because there exists a bijection $\mathbb{N} \to \mathbb{Z}$.
- $|\mathbb{N}| \neq |\mathbb{R}|$ because there exists no bijection $\mathbb{N} \to \mathbb{R}$.

Cardinality of Sets

Definition

Suppose A is a set. Then A is **countably infinite** *if $|\mathbb{N}| = |A|$, that is, if there exists a bijection $\mathbb{N} \to A$. The set A is* **countable** *if it is finite or* **countably infinite**. *The set A is* **uncountable** *if it is infinite and $|\mathbb{N}| \neq |A|$, that is, if A is infinite and there is no bijection $\mathbb{N} \to A$.*

Thus $\mathbb{Z}$ is countably infinite but $\mathbb{R}$ is uncountable.

Definition

The cardinality of the natural numbers is denoted as $\aleph_0$. That is, $|\mathbb{N}| = \aleph_0$. Thus any countably infinite set has cardinality $\aleph_0$.

Example

Let $Ç = \{2k : k \in \mathbb{Z}\}$ *be the set of even integers. The function* $f : \mathbb{Z} \to Ç$ *defined as* $f(n) = 2n$ *is easily seen to be a bijection, so we have* $|\mathbb{Z}| = |Ç|$*. Thus, as* $|\mathbb{Z}| = |\mathbb{N}| = |Ç|$*, the set* $Ç$ *is countably infinite and* $|Ç| = \aleph_0$*.*

Theorem

A set A *is countably infinite if and only if its elements can be arranged in an infinite list* $a_1, a_2, a_3, a_4, \ldots$

Theorem

The set $\mathbb{Q}$ *of rational numbers is countably infinite.*

To prove this, we just need to show how to write the set $\mathbb{Q}$ in list form. Begin by arranging all rational numbers in an infinite array. This is done by making the following chart.

$$
\begin{array}{cccccccccccc}
0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & 5 & -5 & \cdots \\
\hline
\frac{0}{1} & \frac{1}{1} & \frac{-1}{1} & \frac{2}{1} & \frac{-2}{1} & \frac{3}{1} & \frac{-3}{1} & \frac{4}{1} & \frac{-4}{1} & \frac{5}{1} & \frac{-5}{1} & \cdots \\
 & \frac{1}{2} & \frac{-1}{2} & \frac{2}{3} & \frac{-2}{3} & \frac{3}{2} & \frac{-3}{2} & \frac{4}{3} & \frac{-4}{3} & \frac{5}{2} & \frac{-5}{2} & \cdots \\
 & \frac{1}{3} & \frac{-1}{3} & \frac{2}{5} & \frac{-2}{5} & \frac{3}{4} & \frac{-3}{4} & \frac{4}{5} & \frac{-4}{5} & \frac{5}{3} & \frac{-5}{3} & \cdots \\
 & \frac{1}{4} & \frac{-1}{4} & \frac{2}{7} & \frac{-2}{7} & \frac{3}{5} & \frac{-3}{5} & \frac{4}{7} & \frac{-4}{7} & \frac{5}{4} & \frac{-5}{4} & \cdots \\
 & \frac{1}{5} & \frac{-1}{5} & \frac{2}{9} & \frac{-2}{9} & \frac{3}{7} & \frac{-3}{7} & \frac{4}{9} & \frac{-4}{9} & \frac{5}{6} & \frac{-5}{6} & \cdots \\
 & \frac{1}{6} & \frac{-1}{6} & \frac{2}{11} & \frac{-2}{11} & \frac{3}{8} & \frac{-3}{8} & \frac{4}{11} & \frac{-4}{11} & \frac{5}{7} & \frac{-5}{7} & \cdots \\
 & \frac{1}{7} & \frac{-1}{7} & \frac{2}{13} & \frac{-2}{13} & \frac{3}{10} & \frac{-3}{10} & \frac{4}{13} & \frac{-4}{13} & \frac{5}{8} & \frac{-5}{8} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$
	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{3}{11}$	$\frac{-3}{11}$	$\frac{4}{15}$	$\frac{-4}{15}$	$\frac{5}{9}$	$\frac{-5}{9}$	$\cdots$
	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{2}{17}$	$\frac{-2}{17}$	$\frac{3}{13}$	$\frac{-3}{13}$	$\frac{4}{17}$	$\frac{-4}{17}$	$\frac{5}{11}$	$\frac{-5}{11}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Beginning at $\frac{0}{1}$ and following the path, we get an infinite list of all rational numbers:

$0,\ 1,\ \frac{1}{2},\ -\frac{1}{2},\ -1,\ 2,\ \frac{2}{3},\ \frac{2}{5},\ -\frac{1}{3},\ \frac{1}{3},\ \frac{1}{4},\ -\frac{1}{4},\ \frac{2}{7},\ -\frac{2}{7},\ -\frac{2}{5},\ -\frac{2}{3},\ -\frac{2}{1},\ 3,\ \frac{3}{2},\ \ldots$

it follows that $\mathbb{Q}$ is countably infinite, that is, $|\mathbb{Q}| = |\mathbb{N}|$.

Theorem

If A and B are both countably infinite, then so is $A \times B$.

Corollary

Given n countably infinite sets $A_1, A_2, \ldots, A_n$, with $n \geq 2$, the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is also countably infinite.

Theorem

If A and B are both countably infinite, then their union $A \cup B$ is countably infinite.

Comparing Cardinalities:

At this point we know that there are at least two different kinds of infinity. On one hand, there are countably infinite sets such as $\mathbb{N}$, of cardinality $\aleph_0$. Then there is the uncountable set $\mathbb{R}$.

Are there other kinds of infinity beyond these two kinds?

Definition

Suppose A and B are sets.

- $|A| = |B|$ *means there is a bijection* $A \to B$.
- $|A| < |B|$ *means there is an injection* $A \to B$, *but no bijection* $A \to B$.
- $|A| \leq |B|$ *means there is an injection* $A \to B$.

Cardinality of Sets

Theorem

If A is any set, then $|A| < |\mathcal{P}(A)|$.

Theorem

An infinite subset of a countably infinite set is countably infinite.

Theorem

If $U \subseteq A$, and U is uncountable, then A is uncountable

The Cantor-Bernstein-Schröder Theorem:

Theorem

If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. In other words, if there are injections $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.

Example

The intervals $[0, 1)$ and $(0, 1)$ in $\mathbb{R}$ have equal cardinalities.

Surely this fact is plausible, for the two intervals are identical except for the endpoint 0. Yet concocting a bijection $[0, 1) \to (0, 1)$ is tricky.

For a simpler approach, note that $f(x) = \frac{1}{4} + \frac{1}{2}x$ is an injection $[0, 1) \to (0, 1)$.
Also, $g(x) = x$ is an injection $(0, 1) \to [0, 1)$.
The Cantor-Bernstein-Schröder theorem guarantees a bijection $h : [0, 1) \to (0, 1)$, so $|[0, 1)| = |(0, 1)|$.