Computer Simulations

A practical approach to simulation

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Chaotic Systems

Introduction to chaotic systems

A short list of famous examples of chaotic systems:

- Thermal convection in fluids
- Forced Pendulum
- Nonlinear optical devices
- Nonlinear electrical circuits
- Chemical reactions
- Classical many-body systems
- Particle accelerations
- Biological model of population dynamics

Chaotic Systems

Introduction to chaotic systems

- The observed chaotic behaviour in time is,
  - neither due to external souurces of noice
  - nor to an infinite number of degrees of freedom
  - nor to the uncertainity associated with quantum mechanics.
- The actual source of irregularity is the property of the nonlinear system of seperating initially close trajectories exponantially fast in a bounded region of phase space.

Chaotic Systems

Introduction to chaotic systems

- It is practically impossible to predict the long term behaviour of chaotic systems.
- In practice one can only fix their initial conditions with finite accuracy.
- Errors increase exponentially fast since the digits in irrational numbers are irregularly distributed.
- Hence in chaotic systems the trajectory becomes unpredictable.

Chaotic Systems

Introduction to chaotic systems

- All iterative systems with nonlinearity may posses chaotic behaviour.
  - Solution of a differential equation to define a dynamical sytem
  - or an iterative equation represent a path in the phase space.
- A criterium for whether the path is chaotic or not must be defined.
- This criterium must be a measure of how close the next point is.

Chaotic Systems

Introduction to chaotic systems

- There are basically three methods of identifying the chaotic behaviour.
  - Lyapunov exponent
  - Power spectrum
  - Correlation function

Chaotic Systems

Basic Concepts of Chaos

- **Fixed point:** is a point which is invariant under the mapping

$$x^* = f(x^*)$$

- Fixed points are also called critical points or equilibrium points.
- For fixed-point analysis it is necessary to know whether the fixed points of a system are stable against small perturbations or not.

Chaotic Systems

Basic Concepts of Chaos

fixed-point analysis of a function $f(x)$

- If x^* is chaged to $x^* + \varepsilon$, then $f(x^*)$ is changed to

$$f(x^* + \varepsilon) \cong f(x^*) + \varepsilon f'(x^*)$$

  which means that x^* is a
  - stable fixed point if $|f'(x^*) < 1|$,
  - unstable if $|f'(x^*) > 1|$.
  - if if $|f'(x^*) = 1|$, the fixed point is called marginally stable.

Chaotic Systems

Basic Concepts of Chaos

fixed-point analysis of a logistic map:

- Consider the map

$$x = 4\lambda\, x\, (1 - x)$$

,

- this map is called Logistic Map,
- $x^* = 0$ and $x = 1 - 1/(4\lambda)$ are the fixed points.
- Logistic map will be studied in detail later.

Chaotic Systems

Basic Concepts of Chaos

fixed-point analysis of differential equations:

- In case of the differential equations a fixed point is a point where the velocity vector $(y_1^{\bullet}, y_2^{\bullet}, \ldots, y_n^{\bullet})$ vanishes.
- As an example consider the harmonic oscillator which is defined by the Hamiltonian,

$$H = \frac{1}{2}(p^2 + q^2)$$

In this case the phase space is a plane (p, q).

Chaotic Systems

Basic Concepts of Chaos

fixed-point analysis of differential equations:

- The equations of motion are

$$\begin{aligned} q^{\bullet} &= p \\ p^{\bullet} &= -q \end{aligned}$$

and obviously the only fixed point is,

$$(p^*, q^*) = (0, 0)$$

Corresponding oscillator at rest.

Chaotic Systems

Basic Concepts of Chaos

fixed-point analysis of differential equations:

- If the solution is perturbed about this fixed point so that $q \rightarrow q^* + \varepsilon_1$ and $p \rightarrow p^* + \varepsilon_2$, the equation of motion becomes,

$$\dot{\varepsilon}_1 = \varepsilon_2, \qquad \dot{\varepsilon}_2 = \varepsilon_1$$

  Hence,

$$\ddot{\varepsilon}_1 + \varepsilon_1 = \ddot{\varepsilon}_2 + \varepsilon_2 = 0$$

  Therefore perturbation about the fixed point produces ossillations about the fixed point.

Chaotic Systems

Basic Concepts of Chaos

fixed-point analysis of van der Pol equation:

- van der Pol equation

$$x^{\bullet\bullet} + b(x^2 - 1)x^{\bullet} + x = 0 \qquad b > 0$$

- by changing variables $y = x, z = x^{\bullet}$ the equation can be written in the form of two coupled first order differential equations

$$\begin{aligned} y^{\bullet} &= z \\ z^{\bullet} &= -b(y^2 - 1)z - y \end{aligned}$$

- the fixed point is $(z^{\bullet}, y^{\bullet}) = (0, 0) = (z^*, y^*)$

Chaotic Systems

Basic Concepts of Chaos

fixed-point analysis of van der Pol equation:

-
$$y = y^* + \varepsilon_1, \qquad z = z^* + \varepsilon_2$$

- substituting into equations, one finds

$$\ddot{\varepsilon}_1 - b\dot{\varepsilon}_1 + \varepsilon_1 = \ddot{\varepsilon}_2 - b\dot{\varepsilon}_2 + \varepsilon_2 = 0 \qquad b > 0$$

- the perturbationns grow exponentially.

Chaotic Systems

Basic Concepts of Chaos

fixed-point analysis:

- If the system start near the fixed point remains near fixed point this is called stable fixed point.
- All trajectories starting near the fixed point move away from it this is called unstable fixed point.

Chaotic Systems

Attractor

- If there exist trajectories starting from the fixed point that forms a closed loop about the fixed point this closed loop is called attractor.
- The solution of van der Pol equation with $b = 0.1$ is an example of a attractor.

Chaotic Systems

Attractor

- Predictable attractor:
- A fixed point attractor:
- A chaotic attractor:
- Limit cycle:

Chaotic Systems

Attractor

Predictable attractor: represent the behaviour to which a system settles down or is attracted to a point or a looping closed cycle.

A fixed point attractor: the system regardless the initial point always approaches to the same point. Example is a mass attached to end of a spring in a fractional environment. It eventually arrives at an equilibrium point

Chaotic Systems

Attractor

A chaotic attractor: is represented by an unpredictable trajectory where a minute difference in starting position of two initially adjacent points leads to totally uncorrelated position later in time

Limit cycle: A limit cycle is a closed, periodic trajectory "isolated" in the sence that no nearly trajectory is also closed. Limit cycles appear only non-linear,dissipative systems, i.e., non-linear systems with fractional forces. Like fixed points limit cycles may be stable and unstable.

Chaotic Systems - Project I

The Logistic Map

- A one dimensional mapping that has played an important role in the recent developments is the Logistic Map.

$$x_{n+1} = 4\lambda x_n(1 - x_n) \qquad 0 < x_0, \lambda \leq 1$$

- For the logistic map, $f(x) = 4\lambda x(1 - x)$ and the fixed points are the solutions of the equation

$$x^* = 4\lambda x^*(1 - x^*) \;\; \rightarrow \;\; x^* = 0 \;\text{ and }\; x^* = 1 - 1/(4\lambda)$$

Chaotic Systems - Project I

The Logistic Map

- Stability of Logistic Map:

$$[f(x^* + \varepsilon) \approx f(x^*) + \varepsilon f'(x^*)]$$

  since $f'(x) = 4\lambda(1 - 2x)$,

  1. for $\lambda < 1/4$, $x^* = 0$ is a stable fixed point.
  2. for $1/4 \leq \lambda \leq 3/4$ $x^* = 1 - 1/(4\lambda)$ is stable fixed point.
  3. for $3/4 \leq \lambda \leq 1$ the logistic map has no fixed points.

Chaotic Systems - Project I

The Logistic Map

- For $0 \leq \lambda \leq 1/4$ we find that whatever x we start out with between 0 and1, the sequence of iterates $\{x_n\}$ generated by the logistic map converges to $x^* = 0$. The stable fixed point $x^* = 0$ for $\lambda < 1/4$ is therefore an attractor.
- Similarly for $1/4 < \lambda < 3/4$ we find that, regardless of the value of $x_0 (\neq 0.1)$ the sequence $\{x_n\}$ converges to fixed point $x^* = 1 - 1/(4\lambda)$. Thus the stable fixed point $x^* = 1 - 1/(4\lambda)$ is an attractor for $1/4 < \lambda < 3/4$.

Chaotic Systems - Project I

The Logistic Map

- For $3/4 \leq \lambda \leq 1$ the logistic map has no fixed points.
- For $\lambda = 0.76$, after some initial transient that depends on the initial seed x_0, the sequence $\{x_n\}$ settles into a two cycle oscillations $\{0.7306, 0.5984, 0.7...\}$.
- This two-cycle is independent of the seed and thus is an attractor of period two.

Chaotic Systems - Project I

The Logistic Map

- For $x_1^* = 0.7306$ and $x_2^* = 0.5984$

$$x_2^* = f(x_1^*) = f(f(x_2^*)) = f^2(x_2^*)$$
$$x_1^* = f(x_2^*) = f(f(x_1^*)) = f^2(x_1^*)$$

where

$$f^2(x) = f(f(x)) = 16\lambda^2 \left[x - (4\lambda + 1)x^2 + 8\lambda x^3 - 4\lambda x^4\right]$$

is called second iterate of f.
Exercise: show that $x = f(f(x))$ has two solutions $x_1^* = 0.7306$ and $x_2^* = 0.5984$

Chaotic Systems - Project I

The Logistic Map

- Let λ_n be the value of λ at which the n^{th} period doubling biturcation occurs.
- Feigenbaum [M.J.Feigenbaum,J.Staf Phys. 1925(1978);21 669(1979)] has established that the sequence $\{\lambda_n\}$ converges geometrically at a rate given by,

$$\delta = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = 4.6692016091...$$

- δ is universal number.

Chaotic Systems - Project I

The Logistic Map

- The rapid convergence of the sequence of λ_n values allows us to estimate λ_{n+1} fairly accurately from λ_n and λ_{n+1}.
- The sequence $\{\lambda_n\}$ has the limit point $\lambda_\infty = \lambda^* = 0.8924864\ldots$ beyond which the sequence $\{x_n\}$ of iterates of the logistic map appears to be chaotic sequence without any periodicities except for certain windows of λ values. For $\lambda = 0.959$, for instance, a 3-cycle $\{0.9588, 0.1515, 0.4931\}$ appears.

Chaotic Systems - Project I

The Logistic Map

Fundamental period	λ at which it first appears	λ at which it becomes unstable	λ at which all cycles 2^nk become unstable
1	0.25	0.75	0.8925(λ_∞)
6(a)	0.9066	0.9076	0.9082
5(a)	0.9346	0.9353	0.9358
3	0.9571	0.9604	0.9624
5(b)	0.9764	0.9765	0.9766
6(b)	0.984379	0.984399	0.984412
4	0.990025	0.990200	0.990300
6(c)	0.994440	0.994446	0.994450
5(c)	0.997565	0.997575	0.997580

Chaotic Systems - Project I

The Logistic Map

Feigenbaum's δ universality in the sense that

(1) $$\delta = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = 4.6692016091...$$

Chaotic Systems - Project II

Projects:

- The period doubling root to chaos applies to all maps with quadratic maxima.
  - Heron map
  - Rössler attractor
  - Lorentz Atractor
  - The Duffing's oscillator
  - The Volterra-Lotka Model

Chaotic Systems - Project II

Heron map

- The Henon map map is a discrete-time dynamical system.
- It is one of the most studied examples of dynamical systems that exhibit chaotic behavior.
- The Henon map takes a point (x_n, y_n) in the plane and maps it to a new point

Chaotic Systems - Project II

Henon map

- Henon map

(2) $$x_{n+1} = y_n + 1 - Ax_n^2$$

(3) $$y_{n+1} = Bx_n$$

- Depending on the initial seed (x_0, y_0), the sequence (x_n, y_n) either settles on to an attractive set or diverges to infinity.
- The set of all points (x_n, y_n) which converge onto an attractor is called the basis of attraction (of that attractor).

Chaotic Systems - Project II

Henon map

- The Henon map depends on two parameters, a and b, which for the canonical Henon map have values of $a = 1.4$ and $b = 0.3$.
- For the canonical values the Henon map is chaotic.
- For other values of a and b the map may be,
  - chaotic,
  - intermittent,
  - or converge to a periodic orbit.

Chaotic Systems - Project II

Henon map

- For a fixed parameter $B = 0.3$, as parameter A varied a sequence of period doubling bifircations can be observed.
- values of A_n for A at which period doubling bifurcations occur are listed below

period 2^n	A_n	$(A_n - A_{n-1})/(A_{n+1} - A_n)$
2	0.3675	
4	0.9125	4.844
8	1.026	4.3269
16	1.051	4.696
32	1.056536	4.636
64	1.05773083	4.7748
128	1.0579808931	4.6696

Chaotic Systems - Project III

The Rössler attractor

- Otto Rössler designed the Rössler attractor in 1976, but the originally theoretical equations were later found to be useful in modeling equilibrium in chemical reactions.
- The Rössler attractor is the attractor for the Rössler system of non liner equations.

Chaotic Systems - Project III

The Rössler attractor

- The defining equations are:

$$\begin{aligned}\frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + ay \\ \frac{dz}{dt} &= b + z(x - c)\end{aligned}$$

- Rössler studied the chaotic attractor with $a = 0.2$, $b = 0.2$, and $c = 5.7$, though properties of $a = 0.1, b = 0.1$, and $c = 14$ h ave been more commonly used since.

Chaotic Systems - Project IV

The Lorenz attractor

- The Lorenz model has important implications for climate and weather prediction.
- The model is an explicit statement that planetary and stellar atmospheres may exhibit a variety of quasi-periodic regimes that are, although fully deterministic, subject to abrupt and seemingly random change.

Chaotic Systems - Project IV

The Lorenz attractor

- The equations that govern the Lorenz oscillator are:

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= x(\rho-z)-y \\ \frac{dz}{dt} &= xy-\beta z \end{aligned}$$

$\sigma, \rho, \beta > 0$. For $\sigma = 10, \beta = 8/3$ and ρ is varied. The system exhibits chaotic behavior for $\rho = 28$.

Chaotic Systems - Project IV

The Lorenz attractor

```
#include<stdio.h>
int main(){//Lorentz Attractor
  int i=0,N=1000;
  double x0,y0,z0,h,x1,y1,z1;
  double sigma=10.0,beta=8.0/3.0,rho=28.0;
  x0=0.01; y0=1.2; z0=0.3;h=0.01;
  while(i++ < N){
     x1=x0+h*(y0-x0)*sigma;
     y1=y0+h*((rho-z0)*x0-y0);
     z1=z0+h*(x0*y0-beta*z0);
     x0=x1;y0=y1; z0=z1;
     printf("%d %6.5f %6.5f %6.5f\n",i,x0,y0,z0);
    }
}
```

Chaotic Systems - Project V

The Duffing's oscillator

- Duffing's oscillator

$$x^{''} + kx' + x^3 = Bcos(t)$$

- In terms of two coupled first order differantial equations:

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= Bcos(t) - kx' + x^3 \end{aligned}$$

(4)

- $k = 10.1$ and $B = 2, 4, 6, 8, 10, 12, 14, 16$

Chaotic Systems - Project V

The Duffing's oscillator

```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>
/* Duffing's Oscillator

  x'' +  k x' + x^3 = B cos(t)

*/
```

Chaotic Systems - Project V

The Duffing's oscillator

```
//  Duffing's Oscillator
//   x'' +  k x' + x^3 = B cos(t)
double B = 16;     // 2,4,6,8,10,12,14
double k = 10.1;   // Spring constant
//Acceleration is calculated
double ax(double x,double v,double t){
  return(B * cos(t) - k * v - x*x*x);
}
//Velocity calculated
double vx(double x,double v,double t){
  return(v);
}
```

Chaotic Systems - Project V

The Duffing's oscillator

```
int main(){
  double x=1.0,v=0.0; // Initial values of x and v
  double dt=0.01;     // time step
  double t = 0.0;     // time
  double t_end = 100; //  Time limit
  double k1,k2,l1,l2; // Runge-Kutta variables
```

Chaotic Systems - Project V

The Duffing's oscillator

```
printf(" %f %f %f %f \n", t,x,vx);
while(t < t_end){ //If t exceeds the time limit
  // Rumge Kutta 2.
  k1 = ax(x,v,t) * dt;
  l1 = vx(x,v,t) * dt;
  k2 = ax(x+l1,v+k1,t) * dt;
  l2 = ax(x+l1,v+k1,t) * dt;
  // velocity and coordinate at t + dt
  v = v + (k1+k2)/2.0;
  x = x  + (l1+l2)/2.0;
  t = t + dt; // Time increased
  printf(" %f %f %f %f \n", t,x,v); //Print the r
    }return(0);}
```

Chaotic Systems - Project IV

The Volterra-Lotka Model

- The equations that govern the Volterra-Lotka model are

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(b_{12}x_2 - a_1) \\ \frac{dx_2}{dt} &= xx_2(a_2 - b_{21}x_1) \end{aligned}$$

- $x_2 = a_1/b_{12}, \quad x_1 = a_2/b_{21}$ is the unique equilibrium point

Chaotic Systems - Project VI

The Volterra-Lotka Model

- The Volterra-Lotka model is a predatory-prey model.
- Phase space of the model can be studied,

$$\begin{aligned} dx_1 &= x_1(b_{12}x_2 - a_1)dt \\ dx_2 &= xx_2(a_2 - b_{21}x_1)dt \end{aligned}$$

- which implies,

$$\frac{dx_1}{dx_2} = \frac{x_1(b_{12}x_2 - a_1)}{x_2(a_2 - b_{21}x_1)}$$

Chaotic Systems - Project VI

The Volterra-Lotka Model

- This differential equation, can be integrated analytically.
- $$b_{12}x_2 - a_1 \ln x_2 + C = -b_{21}x_1 + a_2 \ln x_1$$

  where C is an arbitrary constant.
- The solution defines a family of closed curves in $x_1 - x_2$ plane.
- Small change in the initial condition results in a small cahange in the final result.

Chaotic Systems - Project VI

The Volterra-Lotka Model

- The model is self limiting; as the prey populatiomn increases, ultimately rate of growth decreases because food is limited.
- Add term in dx_2/dt

$$\frac{dx_2}{dt} = x_2(a_2 - b_{21}x_1 - c_{22}x_2)$$

where c_{22} is the self limiting term.

Chaotic Systems - Project II

Exercises

- Investigate experimentally the mapping $x_{n+1} = \lambda sin \pi x_n$ with x_0 and λ between 0 and 1 as the λ knob is varried.