# Lecture 4: Subgroups

Prof. Dr. Ali Bülent EKİN
Doç. Dr. Elif TAN

Ankara University

# Subgroups

## Definition

Let $(G, \cdot)$ be a group and $\emptyset \neq H \subseteq G$. $(H, \cdot)$ is called a **subgroup** of $G$ (denoted by $H \leq G$) if $H$ is a group with the operation of $G$.

**Examples:**

**1.** Every group has at least two subgroups: $\{e\} \leq G$ and $G \leq G$. The subgroups except $G$ are called the **proper** subgroups and the subgroups except $\{e\}$ are called the **nontrivial** subgroups.

**2.** $(\mathbb{Z}, +) \leq (\mathbb{R}, +)$

**3.** $(M_2(2\mathbb{Z}), \oplus) \leq (M_2(\mathbb{Z}), \oplus)$

**4.** $\left(\left\{\overline{0}, \overline{3}, \right\}, +_4\right) \nleq (\mathbb{Z}_4, +_4)$

**5.** $\left(\left\{\overline{0}, \overline{2}, \overline{4}\right\}, +_6\right) \leq (\mathbb{Z}_6, +_6)$

**6.** Klein-4 group $(V_4 = \{e, a, b, c\}, \cdot)$ has three nontrivial subgroups:

$$\{e, a\} \leq V_4, \quad \{e, b\} \leq V_4, \quad \{e, c\} \leq V_4.$$

Note that $\{e, a, b\} \nleq V_4$,since $ab = c \notin V_4$.

**Theorem (Subgroup Test-1)**

*Let* $(G, \cdot)$ *be a group and* $\emptyset \neq H \subseteq G$.

$$H \leq G \Leftrightarrow \begin{array}{l} (i)\, \forall a, b \in H,\ ab \in H \\ (ii)\, \forall a \in H,\ a^{-1} \in H. \end{array}$$

**Theorem (Subgroup Test-2)**

*Let* $(G, \cdot)$ *be a group and* $\emptyset \neq H \subseteq G$.

$$H \leq G \Leftrightarrow \ \forall a, b \in H,\ ab^{-1} \in H.$$

## Theorem

*Let $(G, \cdot)$ be a group and $\emptyset \neq H \subseteq G$. If $H$ is* **finite** *and* **closed** *under the operation of $G$, then $H \leq G$.*

**Example:** $(Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}, \cdot)$ is a group, called as quaternion group, with multiplication defined as

$$\begin{aligned} i^2 &= j^2 = k^2 = ijk = -1 \\ ij &= k = -ji, jk = i = -kj. \end{aligned}$$

Note that $(Q_8, \cdot)$ is a nonabelian group, since $ij \neq ji$.

- $H_1 = \{\pm 1, \pm i\} \leq Q_8$, since $H_1$ is finite and closed under $\cdot$.
- $H_2 = \{\pm 1, \pm j\} \leq Q_8$, since $H_2$ is finite and closed under $\cdot$.
- $H_3 = \{\pm 1, \pm i, \pm j\} \nleq Q_8$, since $H_3$ is not closed under $\cdot$, $ij = k \notin H_3$.

**Theorem**

*Let $(G, \cdot)$ be a group. If $H_1 \leq G$ and $H_2 \leq G$, then*

1. $H_1 \cap H_2 \leq G$.
2. $H_1 \cup H_2 \nleq G$.

**Example:** Consider the quaternion group. $H_1 = \{\pm 1, \pm i\} \leq Q_8$ and $H_2 = \{\pm 1, \pm j\} \leq Q_8$, but $H_1 \cup H_2 = \{\pm 1, \pm i, \pm j\} \nleq Q_8$.

**Theorem**

*Let $(G, \cdot)$ be a group and $\{H_i\}_{i \in I}$ be a collection of subgroups of $G$. Then*

$$\bigcap_{i \in I} H_i \leq G.$$

**Theorem**

*Let* $(G, \cdot)$ *be a group and let* $H, K \leq G$.

1. $H \cup K \leq G \Leftrightarrow H \subseteq K$ *or* $K \subseteq H$.
2. $HK \leq G \Leftrightarrow HK = KH$ *where*
$$HK = \{hk \mid h \in H, k \in K\}.$$
3. *If* $G$ *is abelian, then* $HK \leq G$.

Note: $HK = KH$ does not mean that their elements are commutative. It means that $h_1 k_1 = k_2 h_2; \exists h_1, h_2 \in H, \exists k_1 k_2 \in K$.

## Definition

Let $(G_1, *_1)$and $(G_2, *_2)$ be any two groups.
$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$ is a group with the operation $*$ defined componentwise:

$$(g_1, g_2) * (h_1, h_2) = (g_1 *_1 h_1, g_2 *_2 h_2)\,.$$

The group $(G_1 \times G_2, *)$ is called the **direct product** of groups $G_1$ and $G_2$.

**Example:** $(\mathbb{Z} \times \mathbb{Z}, +)$ is a commutative group.

## Theorem

*Let $G_1$ and $G_2$ be any two groups and let $H \leq G_1, K \leq G_2$. Then*

$$H \times K \leq G_1 \times G_2.$$

# Subgroups

## Theorem

*Let* $(G, \cdot)$ *be a group.* $M(G) := \{a \in G \mid ag = ga, \text{ for all } g \in G\}$ *is called the* **center** *of the group* $G$.

1. $M(G) \leq G$.
2. $M(G) = G \Leftrightarrow G$ *is a commutative group.*

## Theorem

$(G, \cdot)$ *be a group.* $M_G(a) := \{x \in G \mid ax = xa\}$ *is called the* **centralizer** *of the group* $G$.
$M_G(a) \leq G$.

## Theorem

$$M(G) = \bigcap_{a \in G} M_G(a).$$

**Theorem**

*All subgroups of* $\mathbb{Z}$ *are of form* $n\mathbb{Z}$ *for* $n \in \mathbb{Z}_{\geq 0}$.