Lecture 11: Direct Products

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External and Internal Direct Products

The direct product is used to investigate the structural properties of a group. The general idea of the external and internal direct products can be given as follows:

1. If you have two groups, then you can combine these two groups to construct a new larger group which is called the external direct product.
2. If you have a group and if you can factor it into the smaller groups such that these smaller groups are normal subgroups whose intersection is identity and their product gives the whole group, then the larger group is called the internal direct product of these smaller groups.

Actually these are two different perspectives of looking at the same thing.

External Direct Products

Definition

Let $(G_1, *_1)$and $(G_2, *_2)$ be any two groups. The cartesian product

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$$

is a group with the componentwise operation . defined by

$$(g_1, g_2) . (h_1, h_2) = (g_1 *_1 h_1, g_2 *_2 h_2) .$$

The group $(G_1 \times G_2, .)$ is called the **(external) direct product** of groups G_1 and G_2.

- The identity element is (e_{G_1}, e_{G_2})
- The inverse of element (g_1, g_2) is (g_1^{-1}, g_2^{-1})

Direct Products

This definition can be generalized to more than two groups:

$$G_1 \times G_2 \times \cdots \times G_n = \{(g_1, g_2, \ldots, g_n) \mid g_i \in G_i\} .$$

- $|G_1 \times G_2 \times \cdots \times G_n| = |G_1| . |G_2| \ldots . |G_n|$
- $\circ((g_1, g_2, \ldots, g_n)) = \operatorname{lcm}(\circ(g_1), \circ(g_2), \ldots, \circ(g_n))$
- If $G_1, G_2, \ldots, G_n$ are abelian, then $G_1 \times G_2 \times \cdots \times G_n$ is also abelian.
- If $G_1, G_2, \ldots, G_n$ are cyclic, then $G_1 \times G_2 \times \cdots \times G_n$ need not be cyclic.
  Recall that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic $\Leftrightarrow \gcd(m, n) = 1$.

Examples:

1. Let consider the groups $(\mathbb{Z},+)$ and $(G=\{\pm 1,\pm i\},\cdot)$. Then

$$\mathbb{Z}\times G=\{(x,y)\mid x\in\mathbb{Z}, y\in G\}.$$

The identity of $\mathbb{Z}\times G$ is $(0,1)$ since $(x,y)(0,1)=(x+0,y.1)=(x,y)$.
The inverse of (x,y) is $(-x,y^{-1})$ since $(x,y)(-x,y^{-1})=(0,1)$.

2. Consider the groups $\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}$, and $\mathbb{Z}/6\mathbb{Z}$. Then

$$\begin{aligned} & (\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/5\mathbb{Z})\times(\mathbb{Z}/6\mathbb{Z}) \\ = & \{(x,y,z)\mid x\in\mathbb{Z}/3\mathbb{Z}, y\in\mathbb{Z}/5\mathbb{Z}, z\in\mathbb{Z}/6\mathbb{Z}\}. \end{aligned}$$

$|(\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/5\mathbb{Z})\times(\mathbb{Z}/6\mathbb{Z})|=3.5.6=90$
The identity of $(\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/5\mathbb{Z})\times(\mathbb{Z}/6\mathbb{Z})$ is $(0,0,0)$.

Internal Direct Products

Definition

Let G be a group and $H, K \trianglelefteq G$. Then G is called the **internal direct product** of H and K, denoted by $G = H \otimes K$, if $\forall g \in G$ can be uniquely expressed as $g = hk$, $\exists h \in H, k \in K$.

Theorem

Let G be a group and $H, K \trianglelefteq G$. Then

$$G = H \otimes K \Leftrightarrow \begin{array}{ll} (i) & H \cap K = \{e\} \\ (ii) & HK = G. \end{array}$$

Theorem

Let G be a group and G be an internal direct product of H and K. Then

$$G \simeq H \times K.$$

Remarks:

1. Let G be a group. If $\exists H, K \leq G$ such that

$$\left.\begin{array}{l} (i)\ \ H \cap K = \{e\} \\ (ii)\ \ HK = G \\ (iii)\ \ hk = kh; \forall h \in H, \forall k \in K \end{array}\right. \Rightarrow G = H \otimes K.$$

2. Let G be an **abelian** group. If $\exists H, K \leq G$ such that

$$\left.\begin{array}{l} (i)\ \ H \cap K = \{e\} \\ (ii)\ \ H + K = G. \end{array}\right. \Rightarrow G = H \otimes K.$$

Example: Let

$$\begin{aligned} G &= \mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\} \\ H &= \{\overline{0}, \overline{3}\} \trianglelefteq \mathbb{Z}_6 \\ K &= \{\overline{0}, \overline{2}, \overline{4}\} \trianglelefteq \mathbb{Z}_6. \end{aligned}$$

Since $H \cap K = \{\overline{0}\}$ and $H + K = \mathbb{Z}_6$,

$$\mathbb{Z}_6 = H \otimes K.$$

Also since $H \simeq \mathbb{Z}_2$ and $K \simeq \mathbb{Z}_3$, then

$$\mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3.$$

Example: Let

$$\begin{aligned} G &= \mathbb{Z}_{12} \\ H &= \langle \overline{2} \rangle \trianglelefteq \mathbb{Z}_{12} \\ K &= \langle \overline{3} \rangle \trianglelefteq \mathbb{Z}_{12} \\ H' &= \langle \overline{4} \rangle \trianglelefteq \mathbb{Z}_{12}. \end{aligned}$$

$$\begin{aligned} H + K &= \{h + k \mid h \in H, k \in K\} \\ &= \{\overline{2}x + \overline{3}y \mid x, y \in \mathbb{Z}\} \\ &= \{\overline{1}t \mid t \in \mathbb{Z}\} = \mathbb{Z}_{12}, \end{aligned}$$

But $\mathbb{Z}_{12} \neq H \otimes K$, since $H \cap K = \{\overline{0}, \overline{6}\} \neq \{\overline{0}\}$.

$$\begin{aligned} H + H' &= H \neq \mathbb{Z}_{12} \\ K + H' &= \mathbb{Z}_{12}. \end{aligned}$$