# Lecture 12: Finitely Generated Abelian Groups

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# Finite Abelian Groups

Now we give a complete description of all **finite abelian groups**. Then we will generalize it to the finitely generated abelian groups.

## Theorem (The Fundamental Theorem of Finite Abelian Groups)

*Every finite abelian group $G$ is isomorphic to a direct product of cyclic groups in the form*

$$G \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$$

*where the $p_i$ are prime numbers, not necessarily distinct, and the $r_i$ are positive integers.*

- The direct product is unique except for possible rearrangement of the factors; that is, the prime power $p_i^{r_i}$ are unique.
- If the number of partitions of $r_i$ is $p(r_i)$, then the number of all non isomorphic abelian groups of $G$ is $p(r_1)\,p(r_2)\ldots p(r_k)$.

**Example:** Find all possible abelian groups (up to isomorphism) of order 100.
Since $100 = 2^{\mathbf{2}}5^{\mathbf{2}}$, there exists $p(\mathbf{2})\,p(\mathbf{2}) = 4$ different abelian groups.

1. $\mathbb{Z}_{2^2} \times \mathbb{Z}_{5^2} \cong \mathbb{Z}_{100}$
2. $\mathbb{Z}_{2^2} \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_{20} \times \mathbb{Z}_5$
3. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{5^2} \cong \mathbb{Z}_{50} \times \mathbb{Z}_2$
4. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10} \times \mathbb{Z}_{10}$

- Recall that if $G = \langle a_1, a_2, ..., a_n \rangle$ , then $G$ is called the **finitely generated group.**
- A finite abelian group is always finitely generated abelian group.
- Let $G$ be a finitely generated abelian group generated by $X = \{a_1, a_2, ..., a_k\}$ . Then

$$G = \{n_1 a_1 + n_2 a_2 + ... + n_r a_r \mid n_i \in \mathbb{Z}, 1 \leq i \leq r\} .$$

# Finitely Generated Abelian Groups

## Definition

Let $G$ be an abelian group and $\emptyset \neq X = \{a_1, a_2, ..., a_k\} \subseteq G$. Then $X$ is called a **basis** for $G$ if

1. $G = \langle X \rangle$
2. $X$ is linearly independent.

## Definition

Let $F$ be an abelian group. If $F$ has a finite basis, then $F$ is called a **finitely generated free abelian group**. The number of elements in a basis $F$ is called the **rank** of $F$.

**Theorem**

*Let $G$ be an abelian group. Then the followings are equivalent:*

1. *$G$ has a finite basis.*
2. *$G$ is finite internal direct sum of a family of infinite cyclic subgroups.*
3. $G \cong \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$.

- $F$ is a finitely generated free abelian group
  $\Leftrightarrow$ A finitely generated abelian group $\{0\} \neq F$ is torsion-free.
- $F$ is torsion-free $\Leftrightarrow \forall e \neq a \in F, \circ(a) = \infty$

- Let $G$ be an abelian group. The subgroup

$$T(G) = \{a \in G \mid \circ(a) < \infty\} \leq G$$

is called the **torsion group** of $G$.

- Let $G$ be a finitely generated abelian group. If $G/T(G) \neq \{0\} \Rightarrow G/T(G)$ is a finitely generated free abelian group. That is,

$$G/T(G) \cong F.$$

**Examples:**

**1.** $\mathbb{Z}$ is a free abelian group of rank 1, with basis $\{1\}$.

**2.** Every nonzero subgroup of $\mathbb{Z}$ is finitely generated; that is $n\mathbb{Z} = \langle n \rangle$. Thus every nonzero subgroup of $\mathbb{Z}$ is also free.

**3.** $\mathbb{Q}$ is torsion-free, but not finitely generated.

# The Fundamental Theorem of Finitely Generated Abelian Groups

Let $G$ be a finite or infinite group. Every finitely generated abelian group is the direct product of its torsion subgroup and of a torsion-free subgroup.

**Theorem**

*Let $G$ be a finitely generated abelian group. Then*

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$$

*where $m_i \mid m_{i+1}, 1 \leq i \leq r-1$. The numbers $m_i$ are called the* **torsion coefficients** *of $G$ and the number of factors $\mathbb{Z}$ is called the* **Betti number** *of $G$.*
*If $G$ is finite, then*

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}.$$

**Remark:** The Jordan-Hölder Theorem is a nonabelian generalization of this result.

**Example:** Find the torsion coefficients and Betti number of the group

$$G = \mathbb{Z}_{20} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{15} \times \mathbb{Z}_6.$$

Since $20 = 2^2 5, 15 = 3.5$, and $6 = 2.3$, then

$$\begin{aligned} G &\cong \mathbb{Z}_{2^2} \times \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z} \\ &\cong (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5) \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_5) \times \mathbb{Z} \times \mathbb{Z} \\ &\cong \underbrace{\mathbb{Z}_{30} \times \mathbb{Z}_{60}}_{\textit{torsion coefficients}=30,60} \times \underbrace{\mathbb{Z} \times \mathbb{Z}}_{\textit{betti number}=2} \end{aligned}$$

Note that $30 \mid 60$.