Lecture 7: Ordered Rings

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Ordered Rings

Motivated from the order relation $<$ on the real numbers $\mathbb{R}$ defined by "$a < b \Leftrightarrow b - a$ is positive", now we define the **order** in a ring with unity by determining the **positive** elements.

Definition

Let $(R, +, .)$ be a ring with unity. An **ordered ring** is a ring R together with $\emptyset \neq P \subseteq R$ satisfying the following properties:
(i) **Closure under** $+ : \forall a, b \in P,\ a + b \in P$
(ii) **Closure under** $. : \forall a, b \in P,\ a.b \in P$
(iii)**Trichotomy:** For each $a \in R$, exactly one of the followings holds:

$$a \in P,\ a = 0_R,\ -a \in P.$$

The set P is called the set of **positive** elements of R.

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Theorem

Let R be an ordered ring with the set of positive elements P. Let $<$ (less than) be the relation on R defined by "$a < b \Leftrightarrow (b - a) \in P$" for $a, b \in R$. Then for all $a, b, c \in R$ the followings hold:

1. **Trichotomy:** *Exactly one of the following holds:*
$$a < b, \;\; a = b, \;\; b < a$$
2. **Transitivity:** *If $a < b$ and $b < c \Rightarrow a < c$*
3. **Isotoniticy**: *If $a < b \Rightarrow a + c < b + c$*
If $a < b$ and $0_R < c \Rightarrow ac < bc$ and $ca < cb$.

Remark: By considering this theorem, $>$, $\leq$, $\geq$ are defined as

$$\begin{aligned} b &> a \text{ means } a < b \\ a &\leq b \text{ means either } a < b \text{ or } a = b \\ a &\geq b \text{ means either } b < a \text{ or } b = a. \end{aligned}$$

Remark:

- Converse of this theorem also holds. That is, for a given relation $<$ on a nonzero ring R satisfying trichotomy, transitivity, and isotonicity properties, the set
$$P = \{x \in R \mid 0_R < x\}$$
satisfies the conditions for a set of positive elements.

Properties of ordered rings

1. $\forall 0_R \neq a \in R,\ a^2 \in P.$

For $0_R \neq a \in R$, either $a \in P$ or $-a \in P \overset{closure}{\Rightarrow} a^2 = (-a)^2 \in P$.

2. $\mathrm{Char}(R) = 0$. (Every ordered ring has infinite number of elements).

$1_R = 1_R^2 \in P$ and from the closure property,
$1_R = 1_R + 1_R + \cdots + 1_R \in P$. Thus $n1_R \neq 0_R$.

3. R has no zero divisors.

$$\left.\begin{array}{l} For\ 0_R \neq a \in R, either\ \ a \in P\ \ or\ \ -a \in P \\ For\ 0_R \neq b \in R, either\ \ b \in P\ \ or\ \ -b \in P \end{array}\right\}$$

$\overset{closure}{\Rightarrow}$ either $ab \in P$ or $-ab \in P \overset{trichotomy}{\Rightarrow} ab \neq 0_R$.

Ordered integral domain

Definition

An **ordered integral domain** is an ordered ring that is also an integral domain.

Examples:

1. $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ are ordered rings.

2. The field $\mathbb{C}$ can not be ordered. Since squared of elements of $\mathbb{C}$ must be positive, 1 and $i \in \mathbb{C}$, but $1 = 1^2 \in P$ and $i^2 = -1 \notin P$.

3. $\mathbb{Z}_p$ is not an ordered ring. Since characteristic of finite ring is nonzero, no finite ring can be ordered.

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Definition

Let D be an ordered integral domain and $\emptyset \neq S \subseteq D$. If each nonzero subset of S has a smallest element, then S is called a **well-ordered set**.

Example: From the well-ordering principle, $\mathbb{Z}^+$ is a well-ordered set. But $\mathbb{Z}^-$ is not well-ordered, since there does not exist the smallest element. Thus $\mathbb{Z}$ is not well-ordered.

Theorem

Let D be an ordered integral domain and D^+ (the set of positive elements) be a well-ordered set. Then $D \simeq \mathbb{Z}$.

Proof of synopsis: The smallest element of D^+ is 1_D and $D^+ = \{n1_D \mid n \in \mathbb{Z}^+\}$. Then $D = \{n1_D \mid n \in \mathbb{Z}\}$. Show that $f : \begin{array}{ccc} D & \to & \mathbb{Z} \\ n1_D & \longrightarrow & n \end{array}$ is an isomorphism.

Exercises

Let D be an ordered integral domain. For $a, b, c, d \in D$, prove the followings.

1) $a > b$ and $c < 0_D \Rightarrow ac < bc$

2) $a > b$ and $c > d \Rightarrow a + c < b + d$

3) $a < 0_D$ and $0_D < b \Rightarrow ab < 0_D$

4) $a > 0_D$ and $ab > ac \Rightarrow b > c$

5) $a > b > 0_D \Rightarrow a^2 > b^2$