# Lecture 9: Rings of Polynomials

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# Rings of Polynomials

## Definition

Let $R$ be a ring and $R[x]$ be the set of all infinite formal sums

$$f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

where $a_i \in R$ and $a_i = 0_R$ for all but a finite number of values of $i$. An element of $R[x]$ is called a **polynomial** over $R$.

The symbol $x$ is called an **indeterminate** over $R$, and the $a_i$ are called **coefficients** of $f(x)$.

- The **degree** of $f(x)$, denoted by $degf(x)$, is defined as the largest $i$ such that $a_i \neq 0_R$, and the coefficient $a_i$ is called the **leading coefficient.**
  If $R$ has unity and the leading coefficient $a_i = 1_R$, then $f(x)$ is called a **monic polynomial**.
- If all $a_i = 0_R$ in $f(x)$, then $f(x)$ is called **zero polynomial,** and the degree of zero polynomial is undefined.
- An element of $R$ is called a **constant** polynomial, and the degree of a constant polynomial is 0.

- Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

and

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$$

be two polynomials over $R$. The **addition** and **multiplication** of polynomials $f(x)$ and $g(x)$ are defined by

$$f(x) + g(x) := (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n + \cdots$$

$$f(x).g(x) := c_0 + c_1 x + \cdots + c_n x^n + \cdots, \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

- Two polynomials are defined to be **equal** if and only if $a_i = b_i$ for $i = 0, 1, 2, \ldots$.

For the simplicity, if $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ has $a_k = 0_R$ for $k > n$, we denote $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. We omit any term $0_R x^i$ and write a term $1_R x^k$ as $x^k$.

**Remark:** A polynomial over $R$ can also be defined as an infinite sequence $(a_0, a_1, a_2, \ldots)$ where $a_i \in R$ and $a_k = 0_R$ for all $k$ such that $k > n$. The function $\begin{array}{ccc} R & \longrightarrow & R[x] \\ a & \longrightarrow & (a, 0_R, 0_R, \ldots) \end{array}$ is a monomorphism. Then $R$ is embedded in $R[x]$. Now let denote

$$\begin{aligned} ax^0 &:= (a, 0_R, 0_R, \ldots) \\ ax &:= (0_R, a, 0_R, \ldots) \\ ax^2 &:= (0_R, 0_R, a, \ldots) \\ &\vdots \end{aligned}$$

So

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n := (a_0, a_1, a_2, \ldots, a_n, \ldots).$$

If $R$ has unity $1_R$, then we can consider $x$ as an element of $R[x]$ by $1_Rx$ as $x$. That is, $x := (0_R, 1_R, 0_R, \ldots)$. Thus

$$ax = (a, 0_R, 0_R, \ldots)(0_R, 1_R, 0_R, \ldots) = (0_R, a, 0_R, \ldots).$$

**Theorem**

*The set $R[x]$ is a ring with polynomial addition and multiplication. The ring $(R[x], +, .)$ is called* **ring of polynomials** *over $R$.*

1. *If $R$ is commutative, then so is $R[x]$.*
2. *If $R$ has unity, then so is $R[x]$.*
3. *If $D$ is an integral domain, then so is $D[x]$.*
4. *If $F$ is a field, then $F[x]$ is an integral domain.*

**Remark:** If $F$ is a field, then $F[x]$ is not a field. Since the only invertible elements of $F[x]$ are nonzero constant polynomials.

**Theorem**

*Let $f(x)$ and $g(x)$ be nonzero polynomials in $R[x]$. Then*

$$\begin{aligned} \deg(f(x)g(x)) &\leq \deg f(x) + \deg g(x) \\ \deg(f(x)+g(x)) &\leq \max\{\deg f(x), \deg g(x)\}. \end{aligned}$$

*In particular, if $R$ is an integral domain, then*

$$\deg(f(x)g(x)) = \deg f(x) + \deg g(x).$$

**Example:** Let $f(x) = 2x^2 - 2x + 3$, $g(x) = 3x + 1 \in \mathbb{Z}_6[x]$. Then

$$\begin{aligned} f(x)g(x) &= 2x^2 + x + 3 \\ &\Rightarrow \deg(f(x)g(x)) = 2 < \deg f(x) + \deg g(x) = 3. \end{aligned}$$

**Remark:**

- $\mathbb{Z}_n[x]$ is infinite ring with characteristic $n$.
  In $\mathbb{Z}_2[x]$,
  $$(x+1)^2 = (x+1)(x+1) = x^2+1$$
  and
  $$(x+1)+(x+1) = 0x+0 = 0.$$

# Ring of polynomials with n indeterminates

- The ring $(R[x])[y]$ can be seen as the ring of polynomials in $y$ with coefficients that are polynomials in $x$.
  Thus we consider this ring
  $$R[x_1, x_2, \ldots, x_n]$$
  as a ring of polynomials in the $n$ indeterminates $x_i$ with coefficiens in $R$.