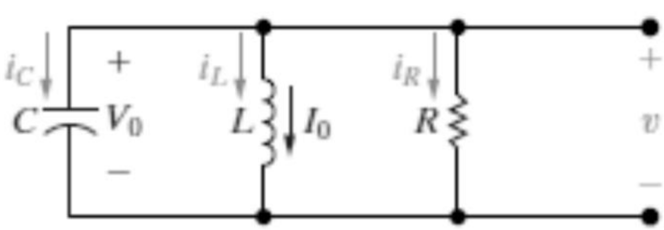
Natural and Step Responses of *RLC* Circuits

Natural response and step response of circuits containing both inductors and capacitors is limited to two simple structures: the parallel *RLC* circuit and the series *RLC* circuit. Finding the natural response of a parallel *RLC* circuit consists of finding the voltage created across the parallel branches by the release of energy stored in the inductor or capacitor or both.



$$\frac{v}{R} + \frac{1}{L}\int_0^t v\,d\tau + I_0 + C\frac{dv}{dt} = 0.$$

$$\frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{d^2v}{dt^2} = 0.$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0.$$

Differential equation describing these circuits is of the second order. Therefore, we sometimes call such circuits **second-order circuits.**

Assume that the solution is of exponential form, that is, to assume that the voltage is of the form

$$v = Ae^{st},$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0.$$

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{Ae^{st}}{LC} = 0,$$

$$Ae^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0,$$

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0.$$

The **characteristic equation** of the differential equation because the roots of this quadratic equation determine the mathematical character of v(t)

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}},$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}.$$

$v = A_1e^{s_1t}$ and

$v = A_2e^{s_2t}$

we can show that their sum also is a solution

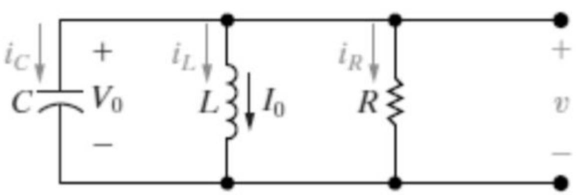
$$v = v_1 + v_2 = A_1e^{s_1t} + A_2e^{s_2t},$$

$$\frac{d^2v}{dt^2} = A_1s_1^2e^{s_1t} + A_2s_2^2e^{s_2t}.$$

$$\frac{dv}{dt} = A_1s_1e^{s_1t} + A_2s_2e^{s_2t},$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0.$$

$$A_1e^{s_1t}\left(s_1^2 + \frac{1}{RC}s_1 + \frac{1}{LC}\right) + A_2e^{s_2t}\left(s_2^2 + \frac{1}{RC}s_2 + \frac{1}{LC}\right) = 0.$$



The natural response of the parallel *RLC* circuit shown

$$v = A_1e^{s_1t} + A_2e^{s_2t}.$$

The roots of the characteristic equation (and) are determined by the circuit parameters *R*, *L*, and *C*. The initial conditions determine the values of the constants A1 and A2

The first step in finding the natural response is to determine the roots of the characteristic equation.

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}},$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}.$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2},$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2},$$

$$\alpha = \frac{1}{2RC},$$

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Analyze the natural response form for each of the three types of damping

When the roots of the characteristic equation are real and distinct, the voltage response of a parallel *RLC* circuit is said to be overdamped.

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t},$$

roots of the characteristic equation

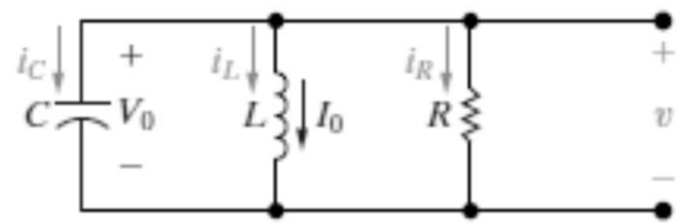
The constants A1 and A2 are determined by the initial conditions, specifically from the values of v(0) and v'(0).

$$i_C(0^+) = \frac{-V_0}{R} - I_0.$$

$$v(0^+) = A_1 + A_2,$$

$$\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2.$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C}.$$



$$
\begin{aligned}
s_1 &= -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} \\
&= -\alpha + j\sqrt{\omega_0^2 - \alpha^2} \\
&= -\alpha + j\omega_d \\
s_2 &= -\alpha - j\omega_d,
\end{aligned}
$$

$$
\omega_d = \sqrt{\omega_0^2 - \alpha^2}.
$$

$$
v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t,
$$

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t},$$

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta.$$

$$= A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

$$= e^{-\alpha t}(A_1 \cos\omega_d t + jA_1 \sin\omega_d t + A_2 \cos\omega_d t - jA_2 \sin\omega_d t)$$

$$= e^{-\alpha t}[\underbrace{(A_1 + A_2)}\cos\omega_d t + \underbrace{j(A_1 - A_2)}\sin\omega_d t].$$

$$v = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

$$= B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t.$$

$$v(0^+) = V_0 = B_1,$$

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = -\alpha B_1 + \omega_d B_2.$$

$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}.$$

$$v = (A_1 + A_2)e^{-\alpha t} = A_0 e^{-\alpha t},$$

cannot satisfy two independent initial conditions with only one arbitrary constant

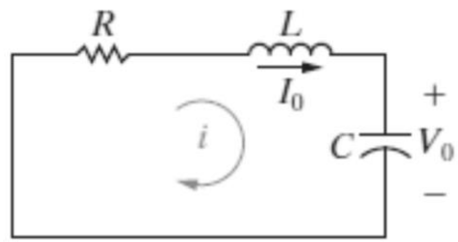
We can trace this dilemma back to the assumption that the solution takes the form of Eq. 8.18.When the roots of the characteristic equation are equal, the solution for the differential equation takes a different form

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}.$$

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t},$$

$$v(0^+) = V_0 = D_2,$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2.$$



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,d\tau + V_0 = 0.$$

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0,$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0.$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0.$$

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}},$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}.$$

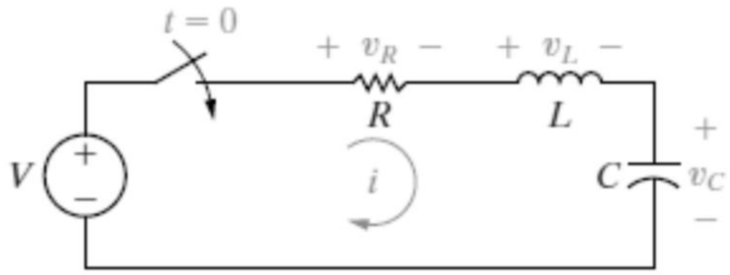
$$\alpha = \frac{R}{2L} \text{ rad/s},$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}.$$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(overdamped)},$$

$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t \quad \text{(underdamped)},$$

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \quad \text{(critically damped)}.$$



$$V = Ri + L\frac{di}{dt} + v_C.$$

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V}{LC}.$$

$$i = C\frac{dv_C}{dt}, \qquad \frac{di}{dt} = C\frac{d^2 v_C}{dt^2}.$$

$$v_C = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t} \quad \text{(overdamped)},$$

$$v_C = V_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t \quad \text{(underdamped)},$$

$$v_C = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} \quad \text{(critically damped)},$$