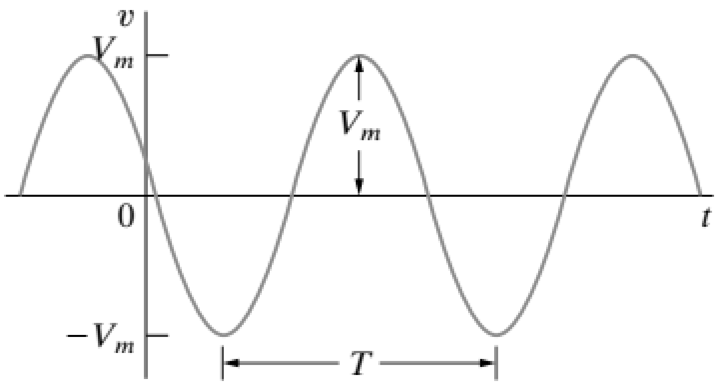
Sinusoidal Steady-State Analysis

The Sinusoidal Source



phase angle

$$v = V_m \cos(\omega t + \phi).$$

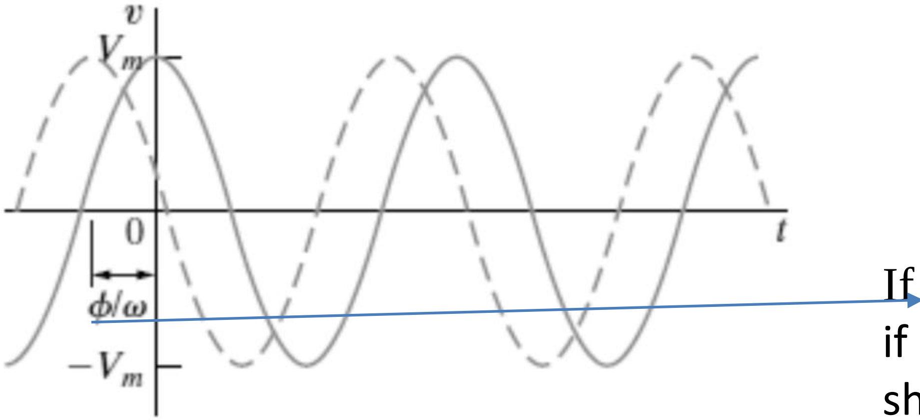
$$f = \frac{1}{T}.$$

A cycle per second is referred to as a hertz, abbreviated Hz. (The term *cycles per second* rarely is used in contemporary technical literature.) The coefficient of *t* in Eq. 9.1 contains the numerical value of *T* or *f*. Omega (w) represents the angular frequency of the sinusoidal function, or

The coefficient gives the maximum amplitude of the sinusoidal voltage.

$$\omega = 2\pi f = 2\pi/T \text{ (radians/second)}.$$

Changing the phase angle shifts the sinusoidal function along the time axis but has no effect on either the amplitude (Vm) or the angular frequency (w)



If φ is negative, the function shifts to the right
if φ is positive, the function shifts to the left
shifted to the right when φ = 0.

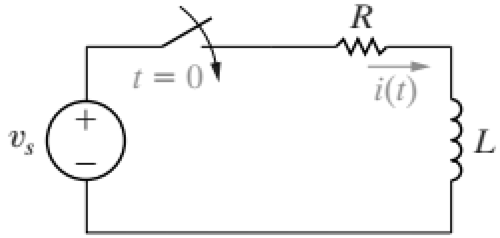
conversion from radians to degrees is

$$\text{(number of degrees)} = \frac{180^\circ}{\pi}\text{(number of radians)}.$$

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)\, dt}.$$

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}}.$$

The Sinusoidal Response



$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi),$$

$$v_s = V_m \cos(\omega t + \phi).$$

$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta),$$

transient component **steady-state component**

We develop a technique for calculating the steady-state response directly, thus avoiding the problem of solving the differential equation. However, in using this technique we forfeit obtaining either the transient component or the total response, which is the sum of the transient and steady state components.

The Phasor

The **phasor** is a complex number that carries the amplitude and phase angle information of a sinusoidal function.1 The phasor concept is rooted in Euler's identity, which relates the exponential function to the trigonometric function:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta.$$

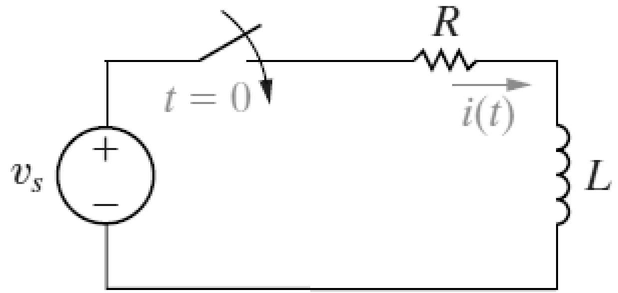
$$\cos\theta = \Re\{e^{j\theta}\},$$

$$\sin\theta = \Im\{e^{j\theta}\},$$

$$\begin{aligned} v &= V_m \cos(\omega t + \phi) \\ &= V_m \Re\{e^{j(\omega t+\phi)}\} \\ &= V_m \Re\{e^{j\omega t} e^{j\phi}\}. \end{aligned}$$

This complex number is by definition the **phasor representation**, or **phasor transform**, of the given sinusoidal function. Thus

$$\mathbf{V} = V_m e^{j\phi} = \mathcal{P}\{V_m \cos(\omega t + \phi)\},$$



We know that the steady-state solution for the current *i is of the form*

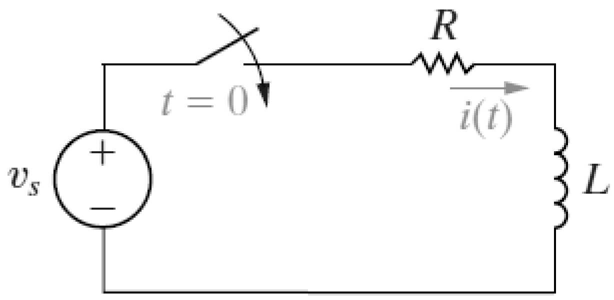
$$i_{ss}(t) = \Re\{I_m e^{j\beta} e^{j\omega t}\},$$

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi),$$

$$\Re\{j\omega L I_m e^{j\beta} e^{j\omega t}\} + \Re\{R I_m e^{j\beta} e^{j\omega t}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}.$$

$$\Re\{(j\omega L + R) I_m e^{j\beta} e^{j\omega t}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}.$$

When both the real and imaginary parts of two complex quantities are equal, then the complex quantities are themselves equal.



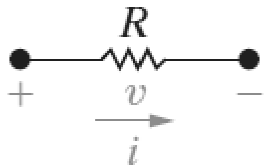
$$(j\omega L + R)I_m e^{j\beta} = V_m e^{j\phi},$$

$$I_m e^{j\beta} = \frac{V_m e^{j\phi}}{R + j\omega L}.$$

The phasor transform, along with the inverse phasor transform, allows you to go back and forth between the time domain and the frequency domain.Therefore, when you obtain a solution, you are either in the time domain or the frequency domain. You cannot be in both domains simultaneously. Any solution that contains a mixture of time domain and phasor domain nomenclature is nonsensical.

The Passive Circuit Elements in the Frequency Domain

The V-I Relationship for a Resistor

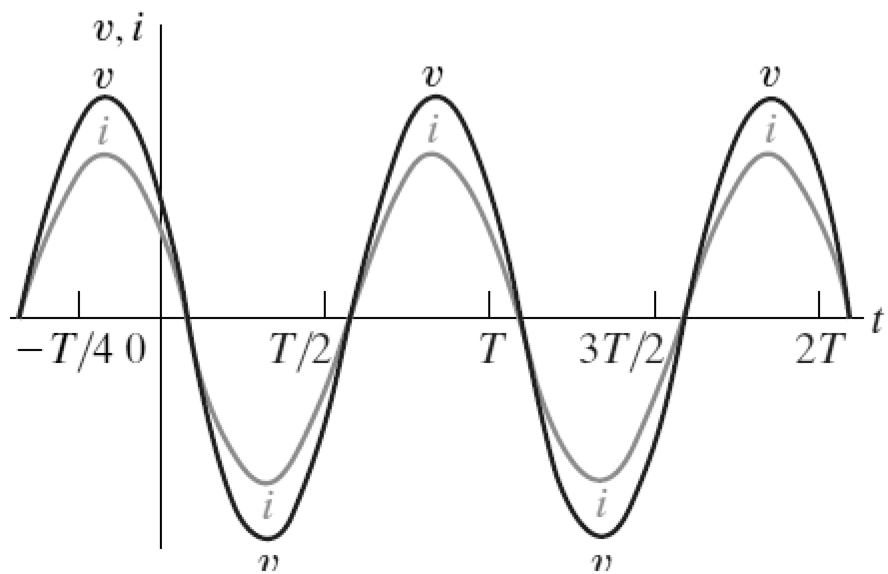


$$v = R[I_m \cos(\omega t + \theta_i)]$$

$$= RI_m[\cos(\omega t + \theta_i)],$$

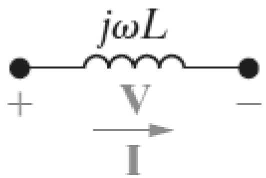
$$\mathbf{V} = RI_m e^{j\theta_i} = RI_m\angle\theta_i.$$

$$\mathbf{V} = R\mathbf{I},$$



The signals are said to be **in phase because they both reach** corresponding values on their respective curves at the same time

The V-I Relationship for an Inductor



$$i = I_m \cos(\omega t + \theta_i),$$

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i).$$

$$v = -\omega L I_m \cos(\omega t + \theta_i - 90^\circ).$$

$$\mathbf{V} = (\omega L \angle 90^\circ) I_m \angle \theta_i$$

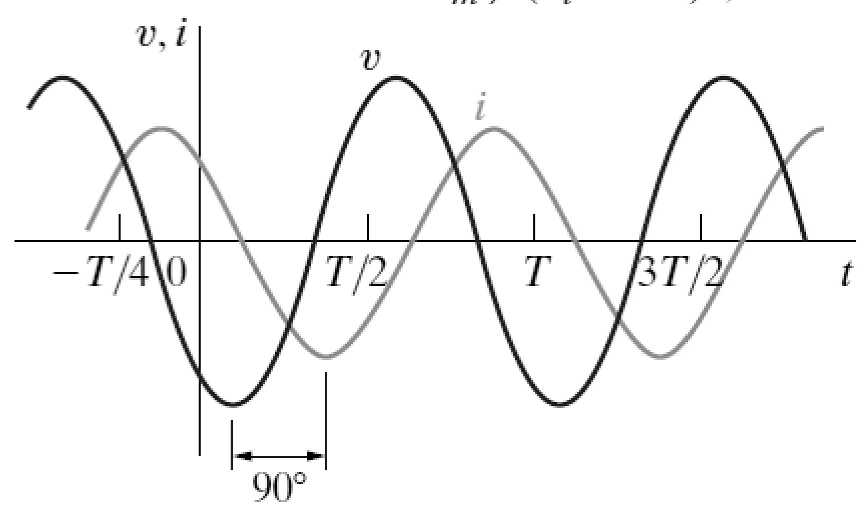
$$\mathbf{V} = -\omega L I_m e^{j(\theta_i - 90^\circ)}$$

$$= -\omega L I_m e^{j\theta_i} e^{-j90^\circ}$$

$$= j\omega L I_m e^{j\theta_i}$$

$$= j\omega L \mathbf{I}.$$

$$= \omega L I_m \angle (\theta_i + 90)^\circ,$$

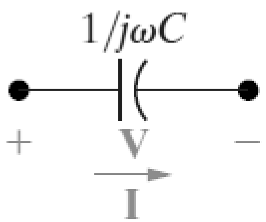


Concept of *voltage leading current or current lagging voltage. For example, the voltage* reaches its negative peak exactly 90 degree before the current reaches its negative peak.

The V-I Relationship for a Capacitor

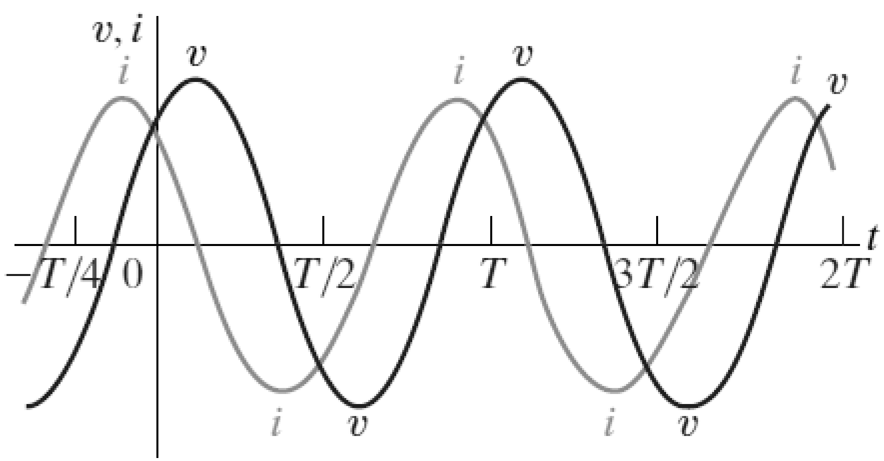
$$i = C\frac{dv}{dt}, \qquad v = V_m \cos(\omega t + \theta_v), \qquad \mathbf{I} = j\omega C\mathbf{V}.$$

$$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}.$$



$$\mathbf{V} = \frac{1}{\omega C}\angle -90^\circ I_m \angle \theta_i^\circ$$

$$= \frac{I_m}{\omega C}\angle (\theta_i - 90)^\circ.$$



the current leads the voltage by 90 degrees.

Kirchhoff’s Laws in the Frequency Domain

Kirchhoff’s Voltage Law in the Frequency Domain

$$v_1 + v_2 + \cdots + v_n = 0,$$

$$V_{m_1} \cos(\omega t + \theta_1) + V_{m_2} \cos(\omega t + \theta_2) + \cdots + V_{m_n} \cos(\omega t + \theta_n) = 0.$$

We now use Euler’s identity

$$\Re\{V_{m_1} e^{j\theta_1} e^{j\omega t}\} + \Re\{V_{m_2} e^{j\theta_2} e^{j\omega t}\} + \cdots + \Re\{V_{m_n} e^{j\theta_n} e^{j\omega t}\}$$

$$\Re\{V_{m_1} e^{j\theta_1} e^{j\omega t} + V_{m_2} e^{j\theta_2} e^{j\omega t} + \cdots + V_{m_n} e^{j\theta_n} e^{j\omega t}\} = 0.$$

$$\Re\{(V_{m_1} e^{j\theta_1} + V_{m_2} e^{j\theta_2} + \cdots + V_{m_n} e^{j\theta_n}) e^{j\omega t}\} = 0,$$

$$\Re\{(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n) e^{j\omega t}\} = 0.$$

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0,$$

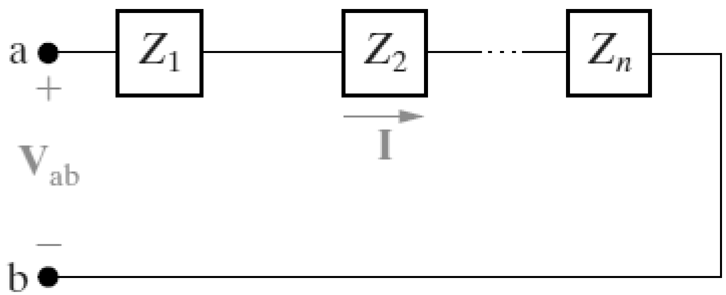
Kirchhoff's Current Law in the Frequency Domain

$$i_1 + i_2 + \cdots + i_n = 0,$$

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0,$$

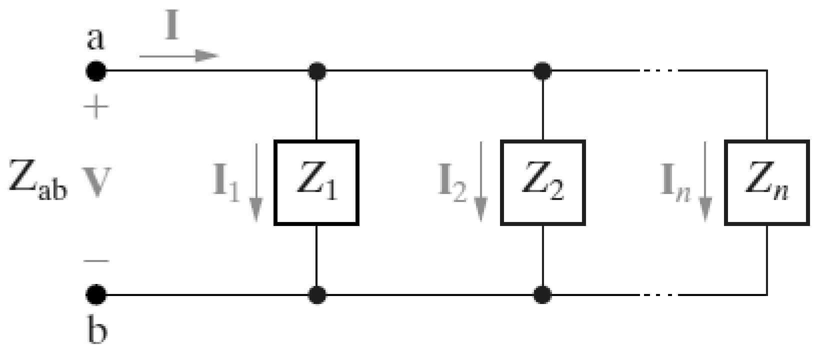
9.6 Series and Parallel Simplifications

Combining Impedances in Series and Parallel



$$\mathbf{V}_{ab} = Z_1\mathbf{I} + Z_2\mathbf{I} + \cdots + Z_n\mathbf{I}$$
$$= (Z_1 + Z_2 + \cdots + Z_n)\mathbf{I}.$$

$$Z_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{I}} = Z_1 + Z_2 + \cdots + Z_n.$$

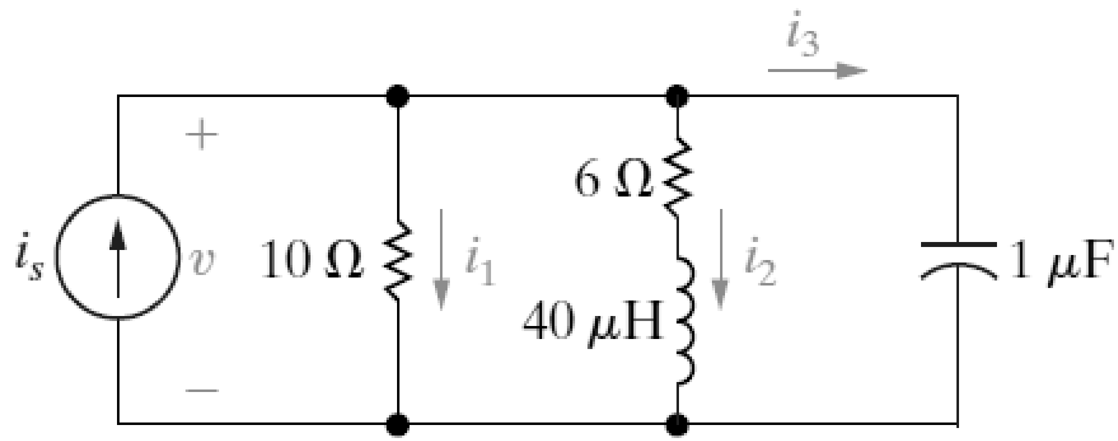


$$\frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}.$$

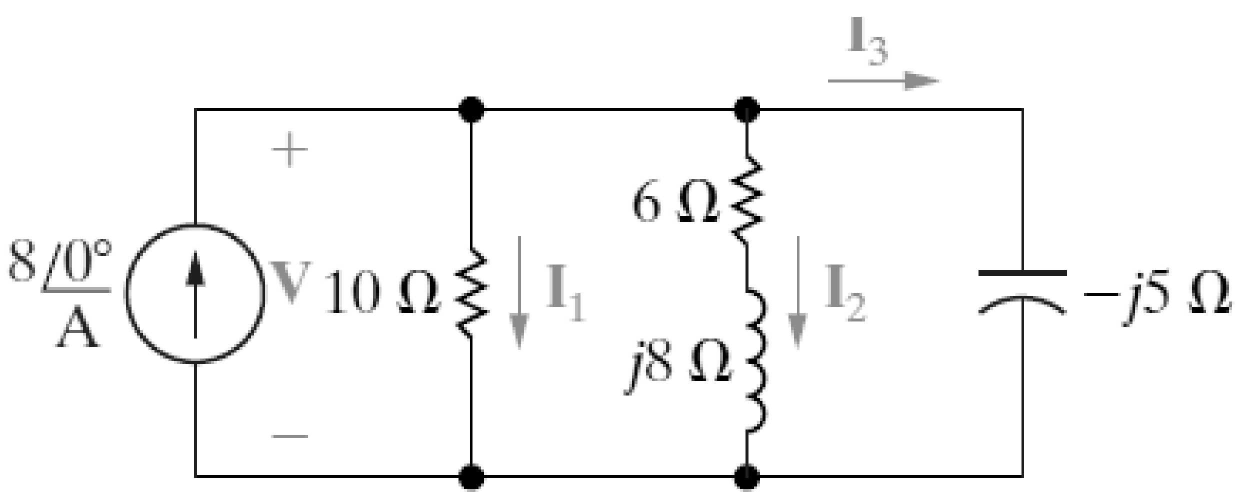
$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n,$$

$$Z_{\mathrm{ab}} = \frac{Z_1 Z_2}{Z_1 + Z_2}.$$

$$\frac{\mathbf{V}}{Z_{\mathrm{ab}}} = \frac{\mathbf{V}}{Z_1} + \frac{\mathbf{V}}{Z_2} + \cdots + \frac{\mathbf{V}}{Z_n}.$$



i_3
i_s
v
+
−
10 Ω
i_1
6 Ω
40 μH
i_2
1 μF



$\mathbf{I}_3$
8/0° A
$\mathbf{V}$
+
−
10 Ω
$\mathbf{I}_1$
6 Ω
$j8$ Ω
$\mathbf{I}_2$
$-j5$ Ω