- Ankara Üniversitesi BLM bölümü

BLM433 Sayısal Analiz Teknikleri

Linear Algebraic Equations

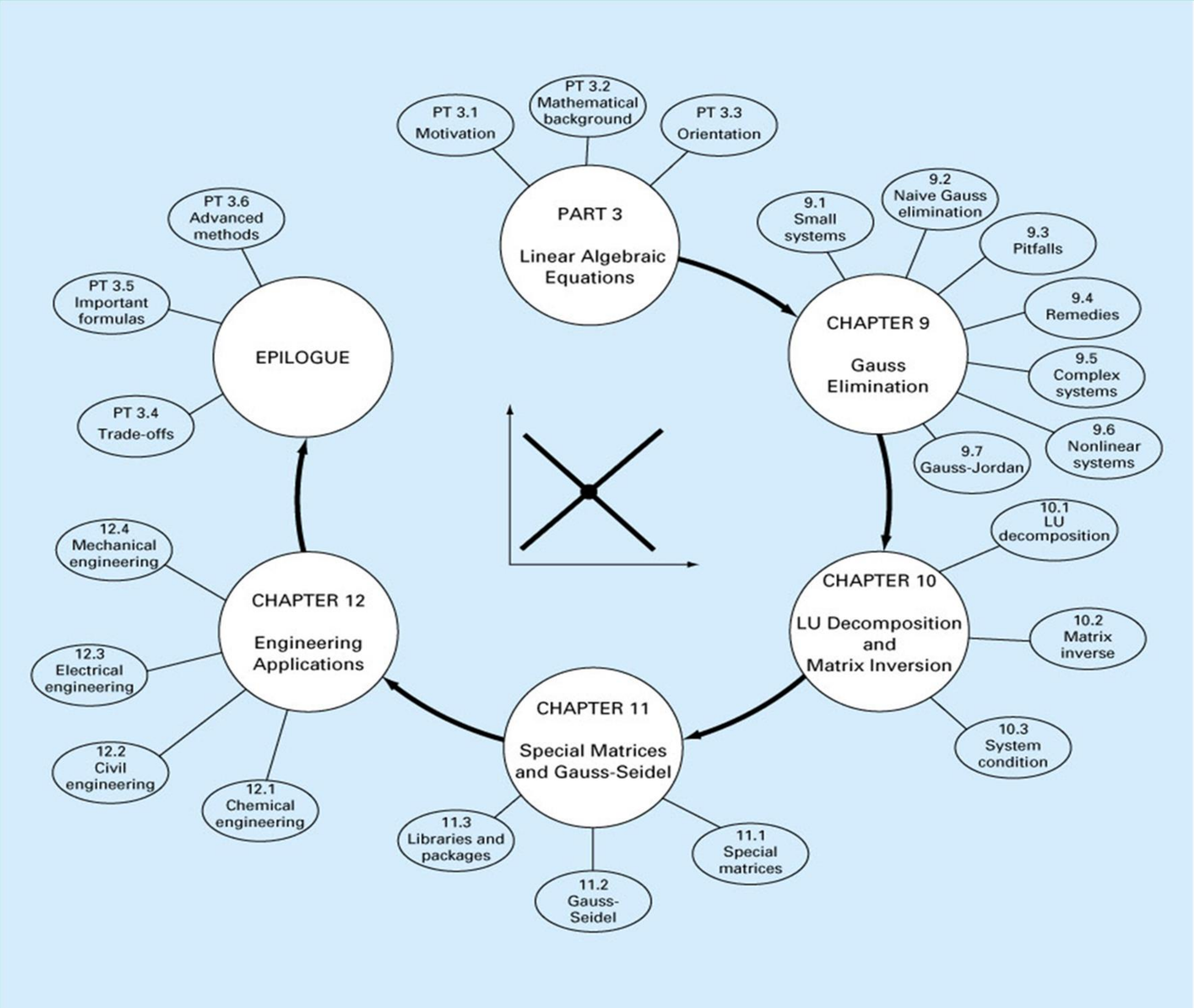
- An equation of the form ax+by+c=0 or equivalently ax+by=-c is called a linear equatic in x and y variables.
- ax+by+cz=d is a linear equation in three variables, x, y, and z.
- Thus, a linear equation in n variables is
- a1x1+a2x2+ … +anxn = b
- A solution of such an equation consists of real numbers c1, c2, c3, … , cn. If you need to wor more than one linear equations, a system of linear equations must be solved simultaneousl

Noncomputer methods

For small number of equations ($n \leq 3$) linear equations can be solved readily by simple techniques such as "method of elimination."

Linear algebra provides the tools to solve such systems of linear equations.

Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical.



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Solving Small Numbers of Equations

There are many ways to solve a system of linear equations:

Graphical method

Cramer's rule

Method of elimination

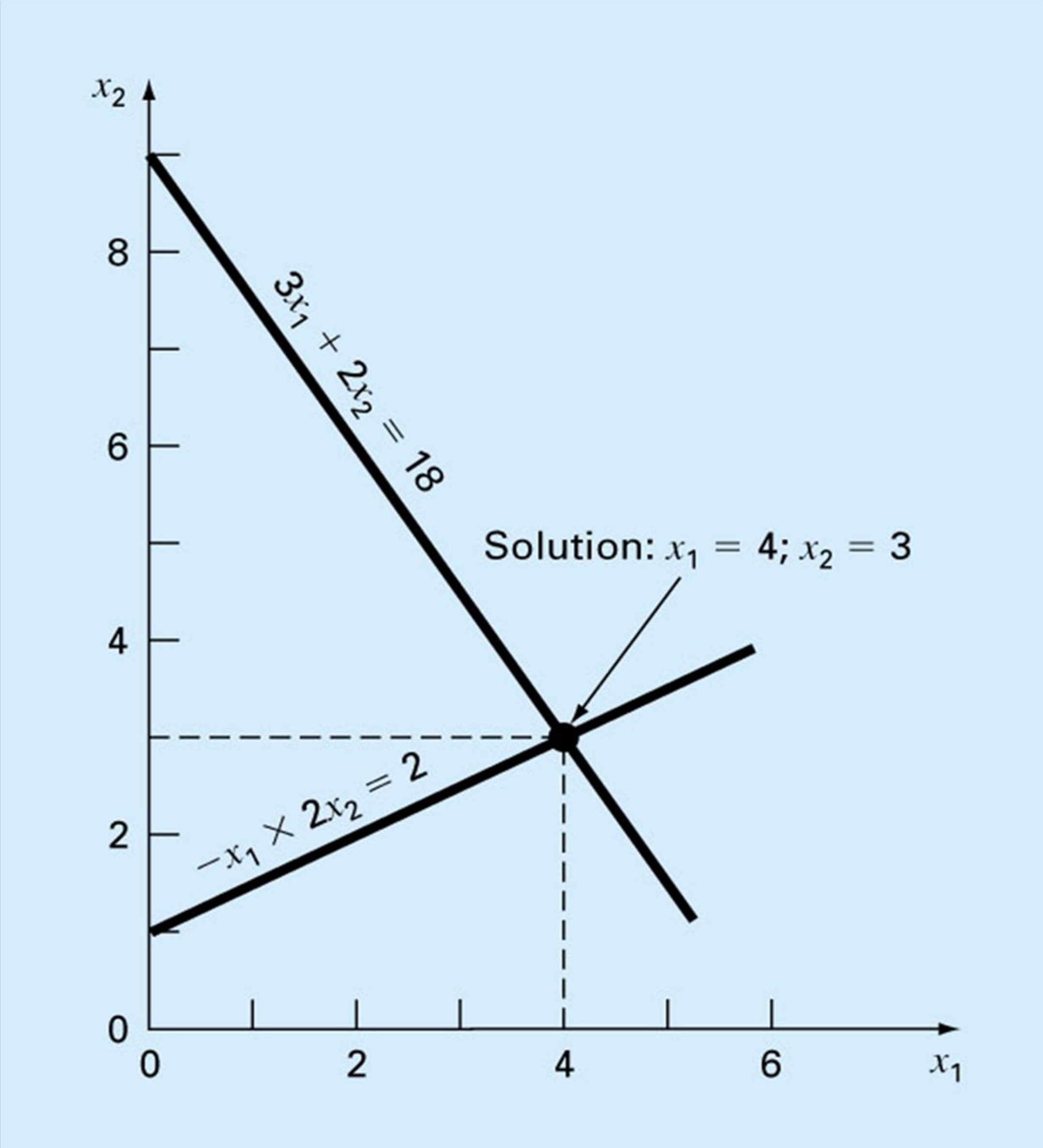
Computer methods

$$a_{11}x_1 + a_{12}x_2 = b_1$$

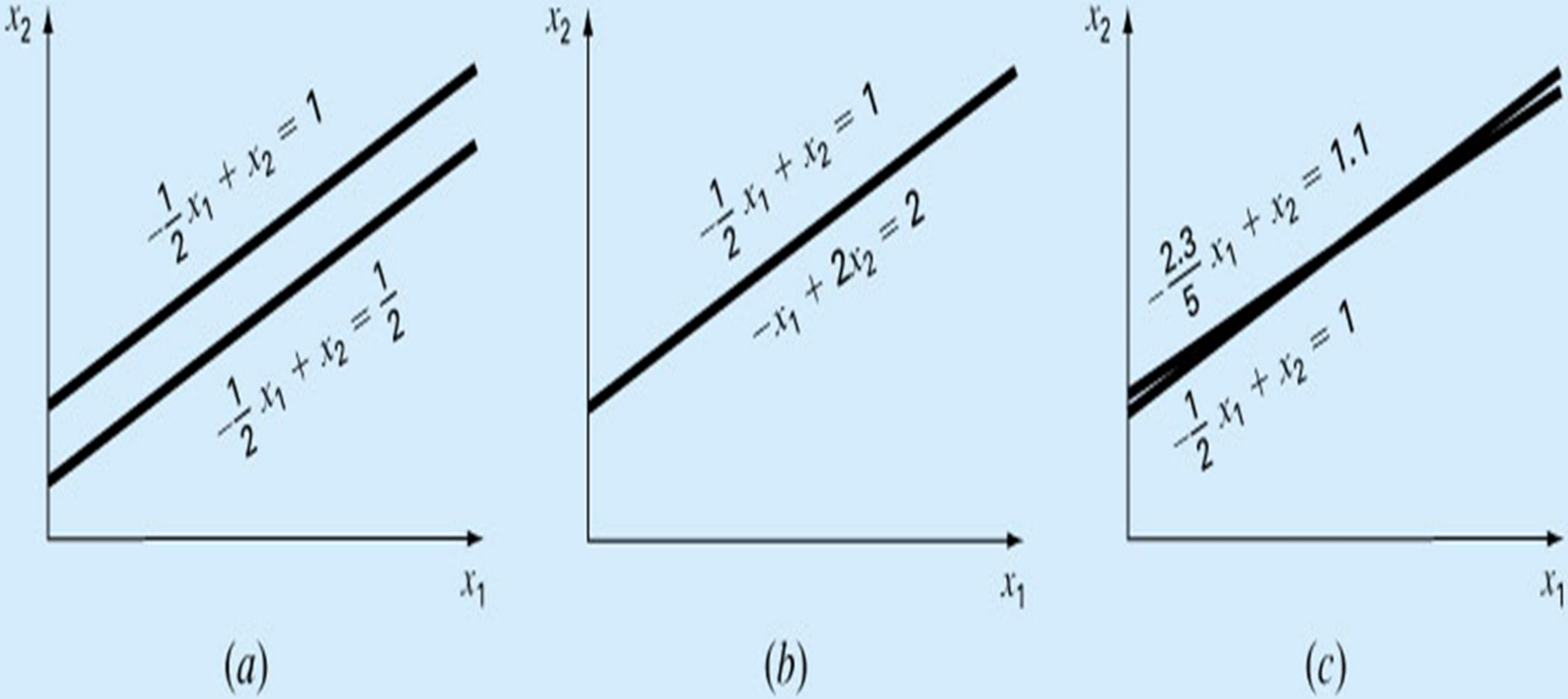
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope})x_1 + \text{intercept}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$



x_2
8
6
4
2
0
0
2
4
6
x_1
$3x_1 + 2x_2 = 18$
Solution: $x_1 = 4; x_2 = 3$
$-x_1 \times 2x_2 = 2$



x_2
$-\frac{1}{2}x_1 + x_2 = 1$
$-\frac{1}{2}x_1 + x_2 = \frac{1}{2}$
x_1
(a)
x_2
$-\frac{1}{2}x_1 + x_2 = 1$
$-x_1 + 2x_2 = 2$
x_1
(b)
x_2
$-\frac{2.3}{5}x_1 + x_2 = 1.1$
$-\frac{1}{2}x_1 + x_2 = 1$
x_1
(c)

Kramers method

$$[A]\{x\}=\{B\}$$

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Assuming all matrices are square matrices, there is a number associated with each square matrix [A] called the determinant, D, of [A]. If [A] is order 1, then [A] has one element:

$$[A]=[a11]$$

$$D=a11$$

For a square matrix of order 3, the minor of an element aij is the determinant of the matrix of order 2 by deleting row i and column j of [A].

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{32}\, a_{23}$$

$$D_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\, a_{33} - a_{31}\, a_{23}$$

$$D_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

Method of elimination

The basic strategy is to successively solve one of the equations of the set for one of the unknowns and to eliminate that variable from the remaining equations by substitution.

The elimination of unknowns can be extended to systems with more than two or three equations; however, the method becomes extremely tedious to solve by hand.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & c_1 \\ a_{21} & a_{22} & a_{23} & \vdots & c_2 \\ a_{31} & a_{32} & a_{33} & \vdots & c_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & c_1 \\ & a'_{22} & a'_{23} & \vdots & c'_2 \\ & & a''_{33} & \vdots & c''_3 \end{bmatrix}$$

Forward elimination

$$\Downarrow$$

$$x_3 = c''_3 / a''_{33}$$

$$x_2 = (c'_2 - a'_{23} x_3) / a'_{22}$$

$$x_1 = (c_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$$

Back substitution

Division by zero. It is possible that during both elimination and back-substitution phases a division by zero can occur.

Round-off errors.

Ill-conditioned systems. Systems where small changes in coefficients result in large changes in the solution. Alternatively, it happens when two or more equations are nearly identical, resulting a wide ranges of answers to approximately satisfy the equations. Since round off errors can induce small changes in the coefficients, these changes can lead to large solution errors.

Singular systems. When two equations are identical, we would loose one degree of freedom and be dealing with the impossible case of n-1 equations for n unknowns. For large sets of equations, it may not be obvious however. The fact that the determinant of a singular system is zero can be used and tested by computer algorithm after the elimination stage. If a zero diagonal element is created, calculation is terminated.

Use of more significant figures.

Pivoting. If a pivot element is zero, normalization step leads to division by zero. The same problem may arise, when the pivot element is close to zero. Problem can be avoided:

Partial pivoting. Switching the rows so that the largest element is the pivot element.

Complete pivoting. Searching for the largest element in all rows and columns then switching

Gauss Jordan elimination

It is a variation of Gauss elimination. The major differences are:

When an unknown is eliminated, it is eliminated from all other equations rather than just the subsequent ones.

All rows are normalized by dividing them by their pivot elements.

Elimination step results in an identity matrix.

Consequently, it is not necessary to employ back substitution to obtain solution.