Ankara Ü. BLM bölümü

BLM 433 Sayısal Analiz Teknikleri

Interpolation

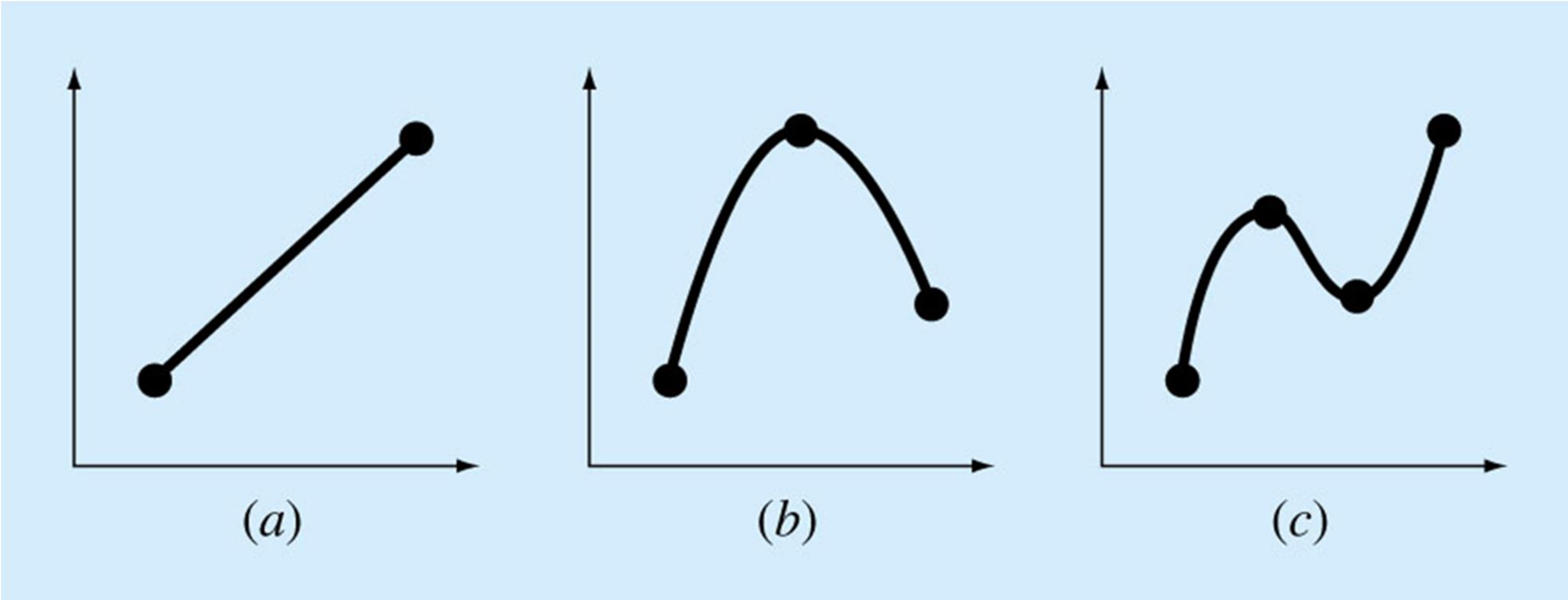
Estimation of intermediate values between precise data points. The most common method is:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Although there is one and only one nth-order polynomial that fits n+1 points, there are a variety of mathematical formats in which this polynomial can be expressed:

The Newton polynomial

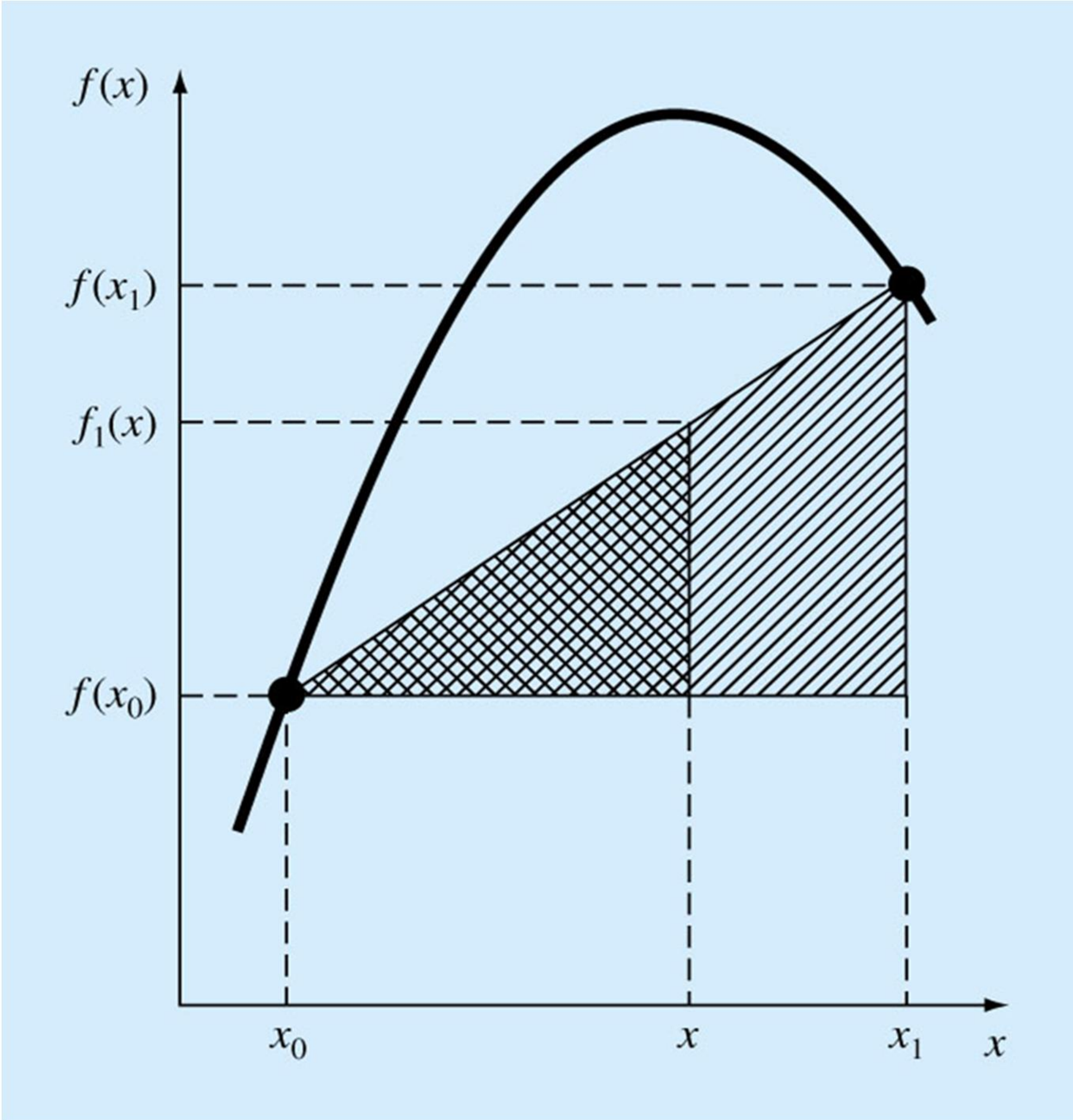
The Lagrange polynomial


(a)
(b)
(c)

Newton divided-difference method

$$\frac{f_1(x)-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x-x_0}$$

$$f_1(x)=f(x_0)+\frac{f(x_1)-f(x_0)}{x-x_0}(x-x_0)$$



$f(x)$
$f(x_1)$
$f_1(x)$
$f(x_0)$
x_0
x
x_1
x

Quadratic interpolation

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$x = x_0 \qquad b_0 = f(x_0)$

$x = x_1 \qquad b_1 = \dfrac{f(x_1) - f(x_0)}{x - x_0}$

$x = x_2 \qquad b_2 = \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$

$$f_n(x) = f(x_0) + (x - x_0) f[x_1, x_0] + (x - x_0)(x - x_1) f[x_2, x_1, x_0]$$

$$+ \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f[x_n, x_{n-1}, \cdots, x_0]$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

$$b_n = f[x_n, x_{n-1}, \cdots, x_1, x_0]$$

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$\vdots$$

$$f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

Errors estimation

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

$$R_n \cong f[x_{n+1}, x_n, x_{n-1}, \ldots, x_0](x-x_0)(x-x_1)\cdots(x-x_n)$$

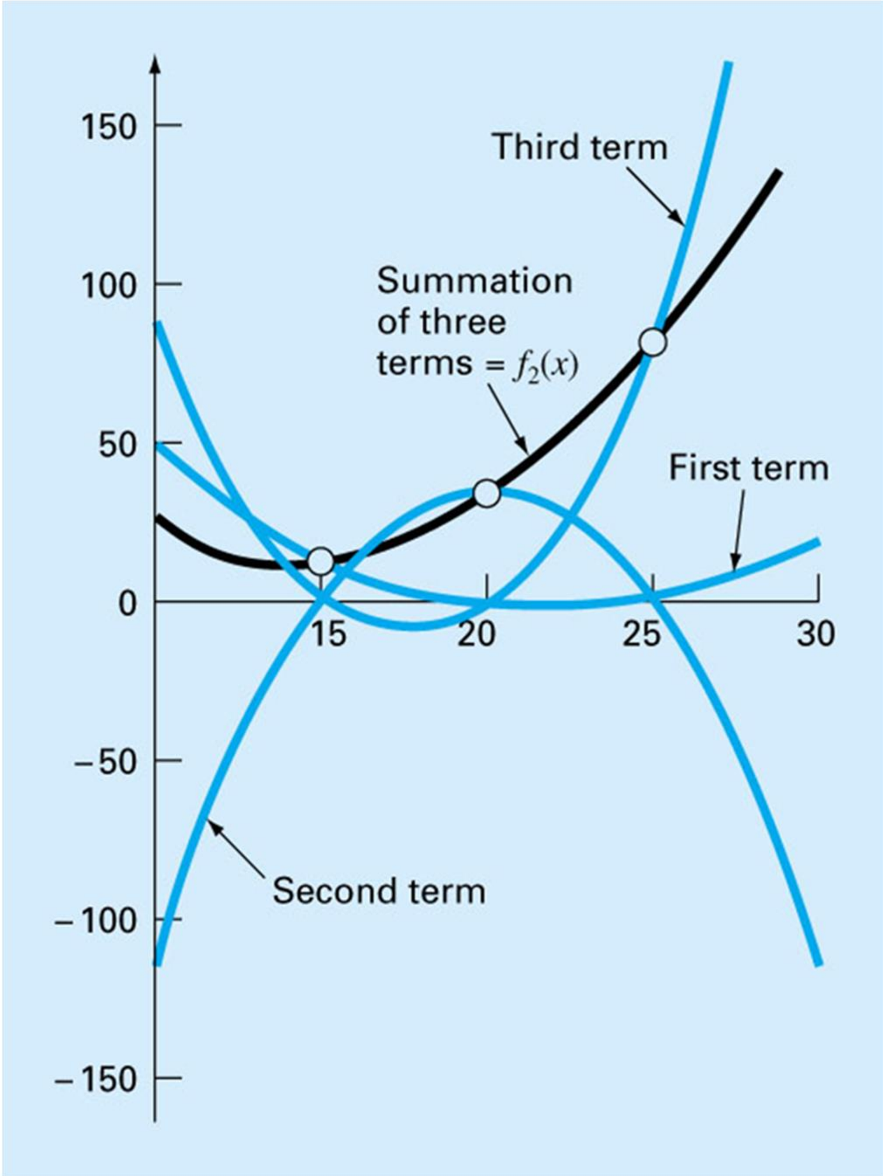
Lagrange Interpolating Polynomials

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$



$$R_n = f[x, x_n, x_{n-1}, \ldots, x_0] \prod_{i=0}^{n} (x - x_i)$$

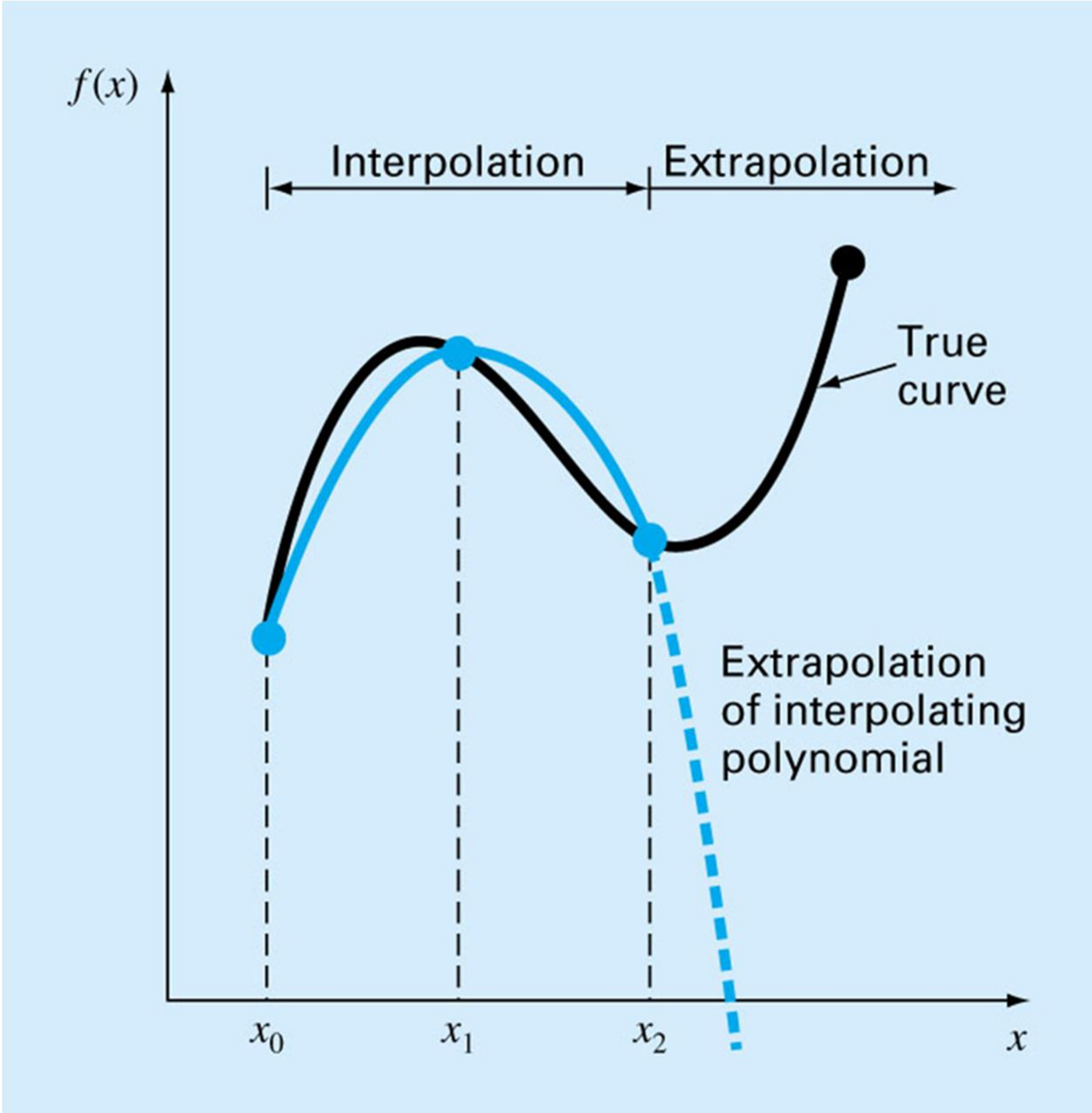
$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_x x^n$$

$$f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 \cdots + a_n x_0^n$$

$$f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 \cdots + a_n x_1^n$$

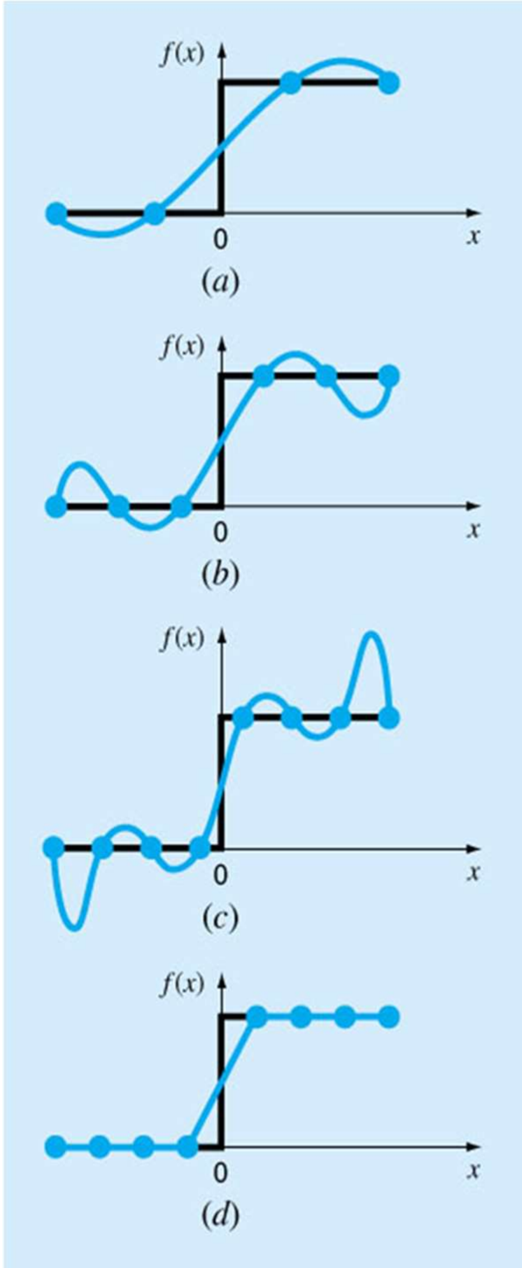
$$\vdots$$

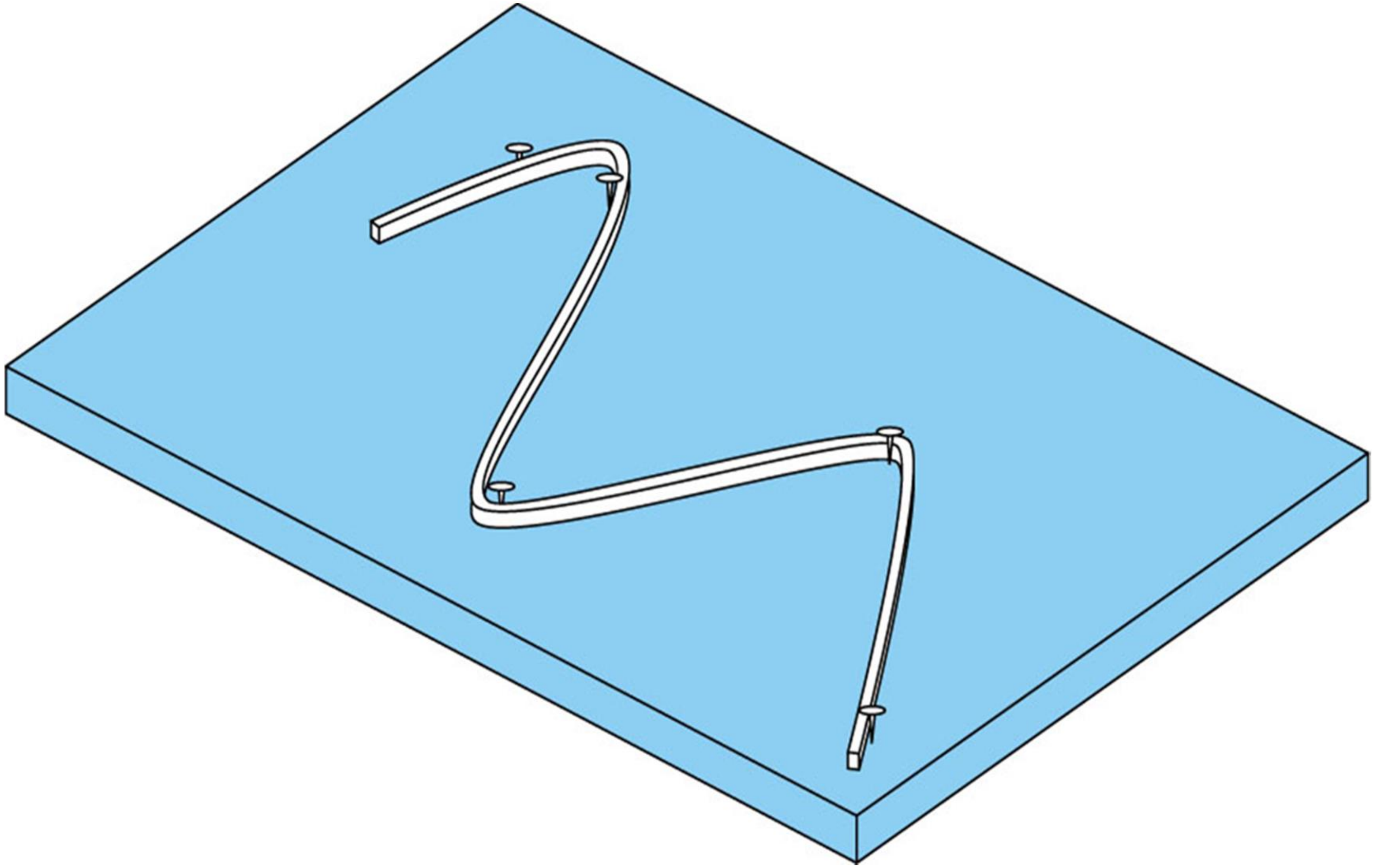
$$f(x_n) = a_0 + a_1 x_n + a_2 x_n^2 \cdots + a_n x_n^n$$

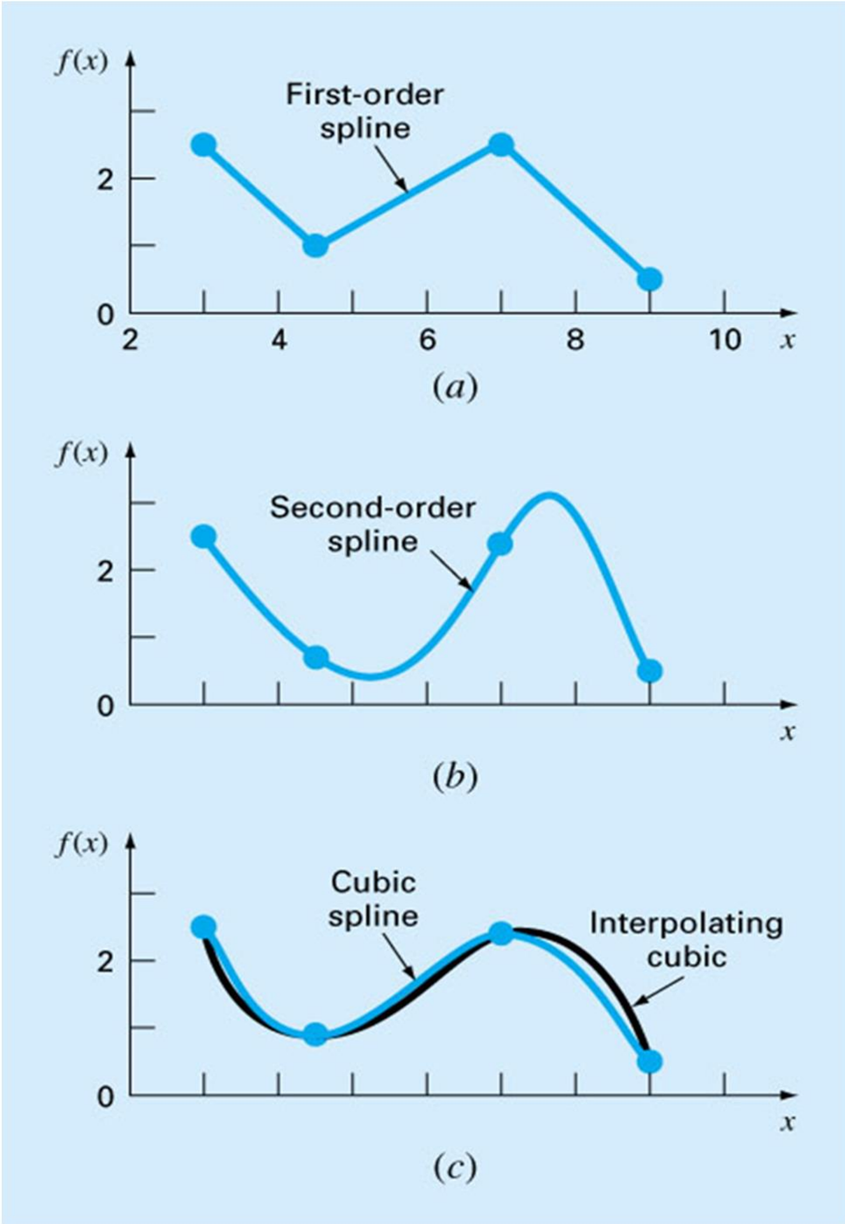

$f(x)$
Interpolation
Extrapolation
True
curve
Extrapolation
of interpolating
polynomial
x_0
x_1
x_2
x

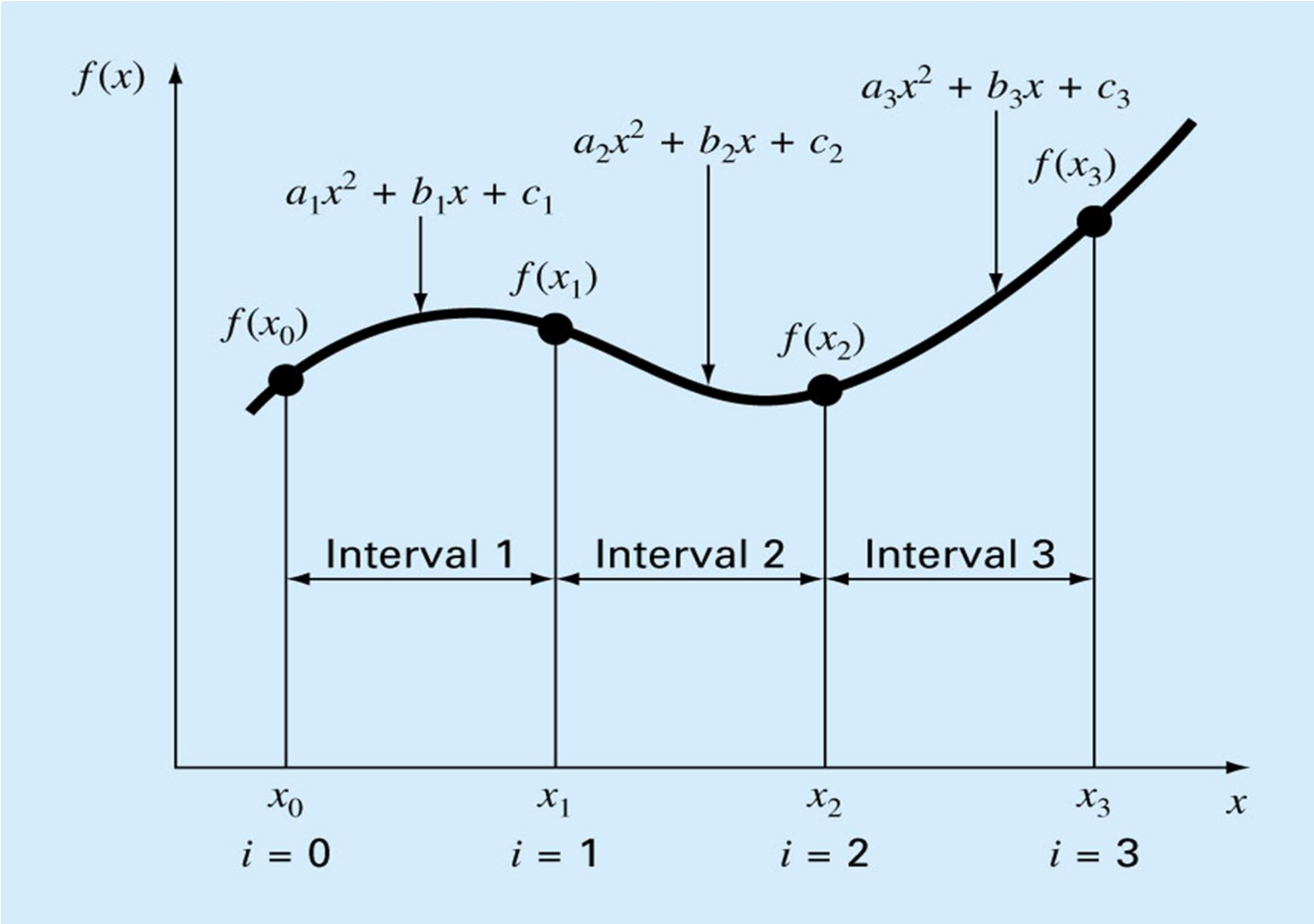
There are cases where polynomials can lead to erroneous results because of round off error and overshoot.

Alternative approach is to apply lower-order polynomials to subsets of data points. Such connecting polynomials are called spline functions.


f(x)
0
x
(a)
f(x)
0
x
(b)
f(x)
0
x
(c)
f(x)
0
x
(d)




f(x)
First-order
spline
2
0
2
4
6
8
10
x
(a)
f(x)
Second-order
spline
2
0
x
(b)
f(x)
Cubic
spline
Interpolating
cubic
2
0
x
(c)



$f(x)$
$a_3x^2 + b_3x + c_3$
$a_2x^2 + b_2x + c_2$
$f(x_3)$
$a_1x^2 + b_1x + c_1$
$f(x_1)$
$f(x_0)$
$f(x_2)$
Interval 1
Interval 2
Interval 3
x_0
x_1
x_2
x_3
x
$i = 0$
$i = 1$
$i = 2$
$i = 3$