# Ankara Ü., BLM bölümü

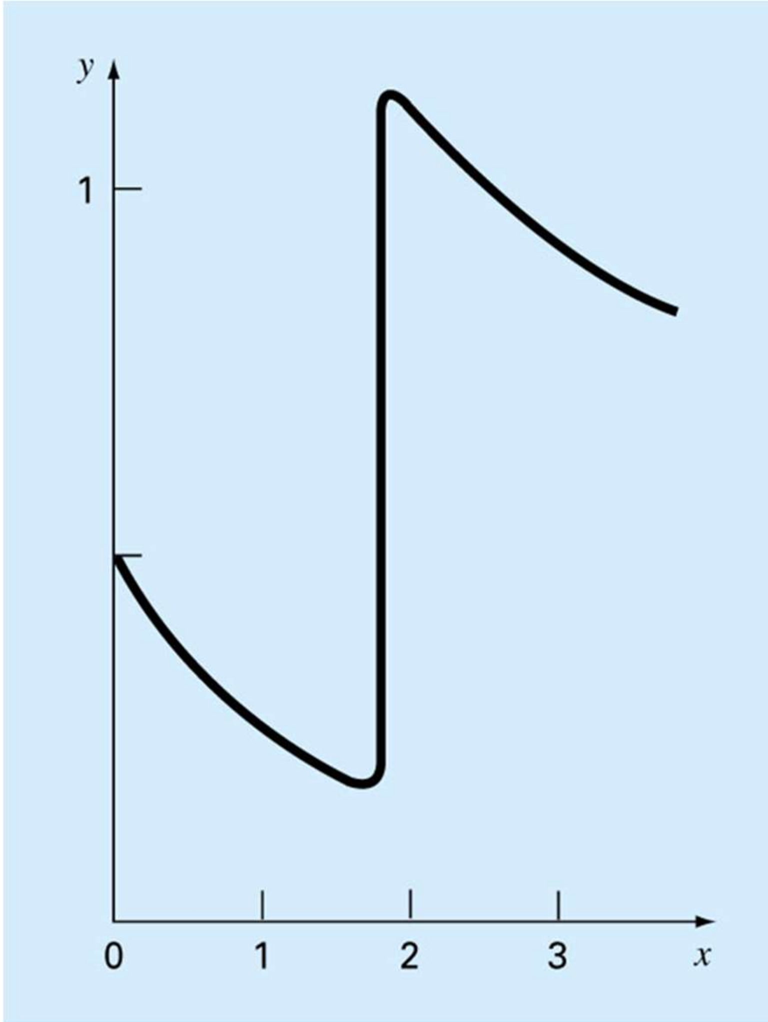
BLM433 Sayısal Analiz Teknikleri

# System of ODE

Many practical problems in engineering and science require the solution of a system of simultaneous ordinary differential equations rather than a single equation:

$$
\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y_1, y_2, \ldots, y_n) \\
\frac{dy_2}{dx} &= f_2(x, y_1, y_2, \ldots, y_n) \\
&\vdots \\
\frac{dy_n}{dx} &= f_n(x, y_1, y_2, \ldots, y_n)
\end{aligned}
$$

For an ODE with an abrupt changing solution, a constant step size can represent a serious limitation.



# Step size control

The strategy is to increase the step size if the error is too small and decrease it if the error is too large. Press et al. (1992) have suggested the following criterion to accomplish this:

$$h_{new} = h_{present} \left| \frac{\Delta_{new}}{\Delta_{present}} \right|^{\alpha}$$

Two areas are covered:

Stiff ODEs will be described – ODEs

that have both fast and slow

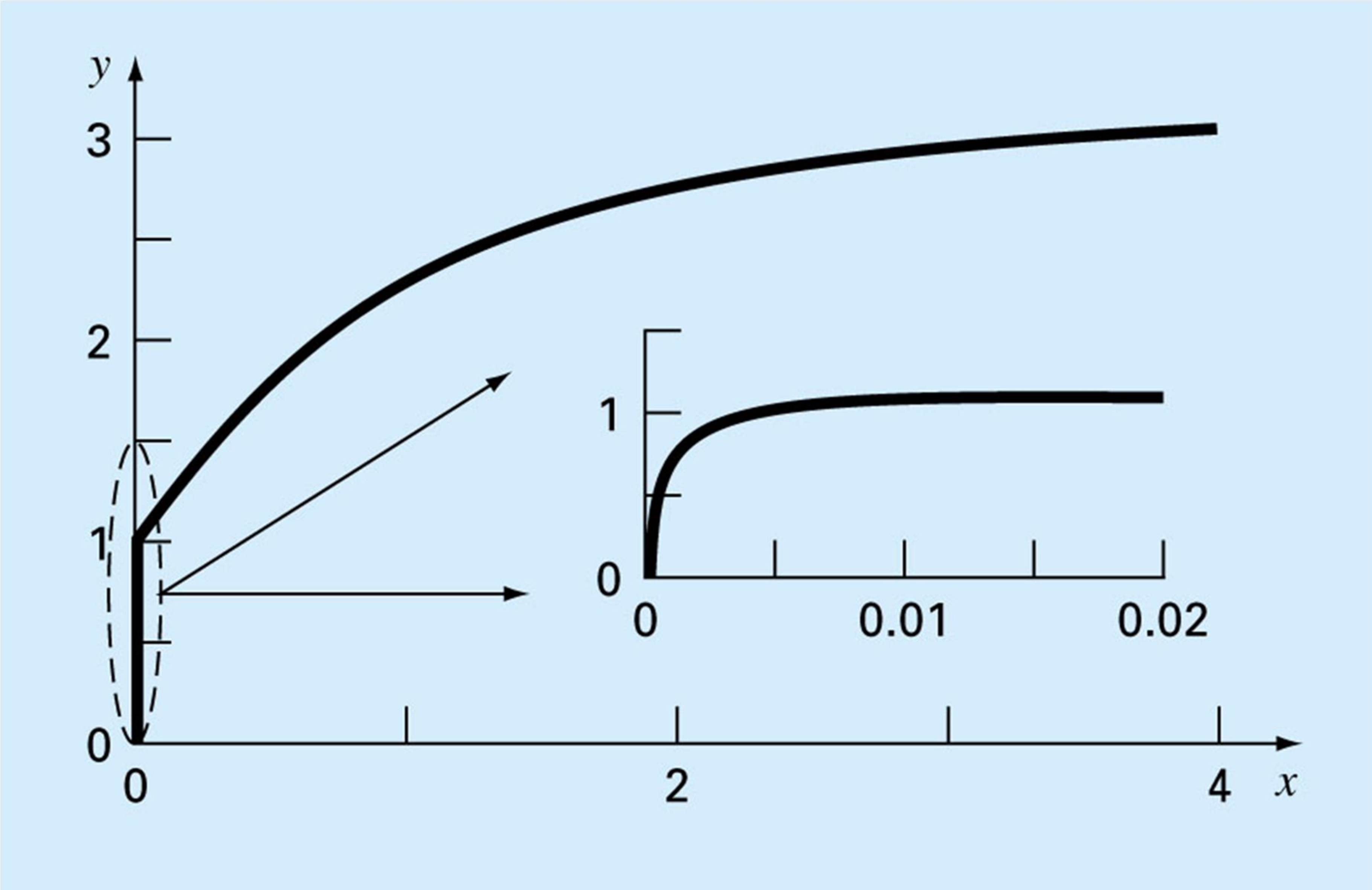
components to their solution.

Implicit solution technique and

multistep methods will be described.

$$\frac{dy}{dt} = -1000y + 3000 - 2000e^{-t}$$

$$y = 3 - 0.998e^{-1000t} - 2.002^{e-t}$$


y
3
2
1
0
0
2
4
x
1
0
0
0.01
0.02

$$\frac{dy}{dt} = -ay$$

$$y = y_0 e^{-at}$$

$$y_{i+1} = y_i + \frac{dy_i}{dt} h$$

$$y_{i+1} = y_i - a y_i h \qquad \text{or} \qquad y_{i+1} = y_i (1 - ah)$$

$$|1 - ah| \prec 1$$

$$h \succ 2/a \quad \Rightarrow \quad |y_i| \to \infty \quad \text{as} \quad i \to \infty$$

Thus, for transient part of the equation, the step size must be <2/1000=0.002 to maintain stability.

While this criterion maintains stability, an even smaller step size would be required to obtain an accurate solution.

Rather than using explicit approaches, implicit methods offer an alternative remedy.

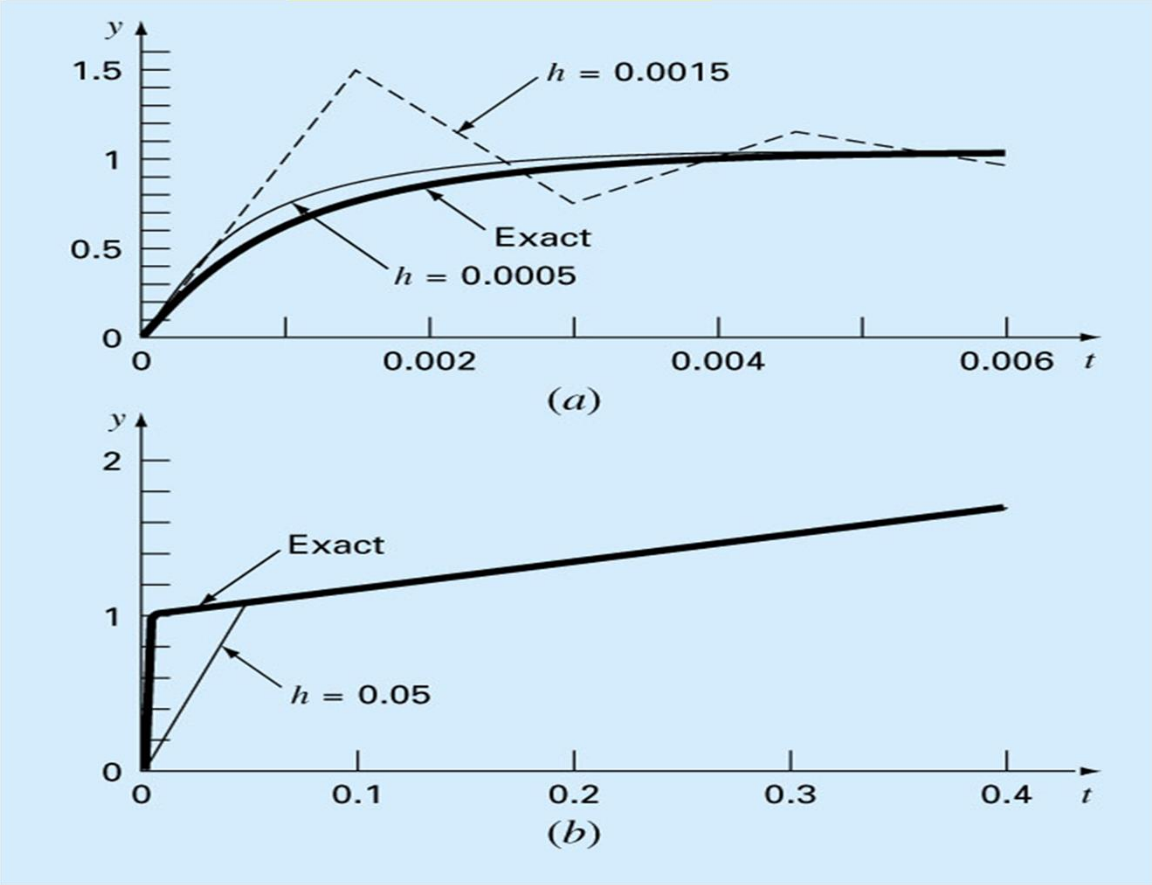
An implicit form of Euler’s method can be developed by evaluating the derivative at a future time.

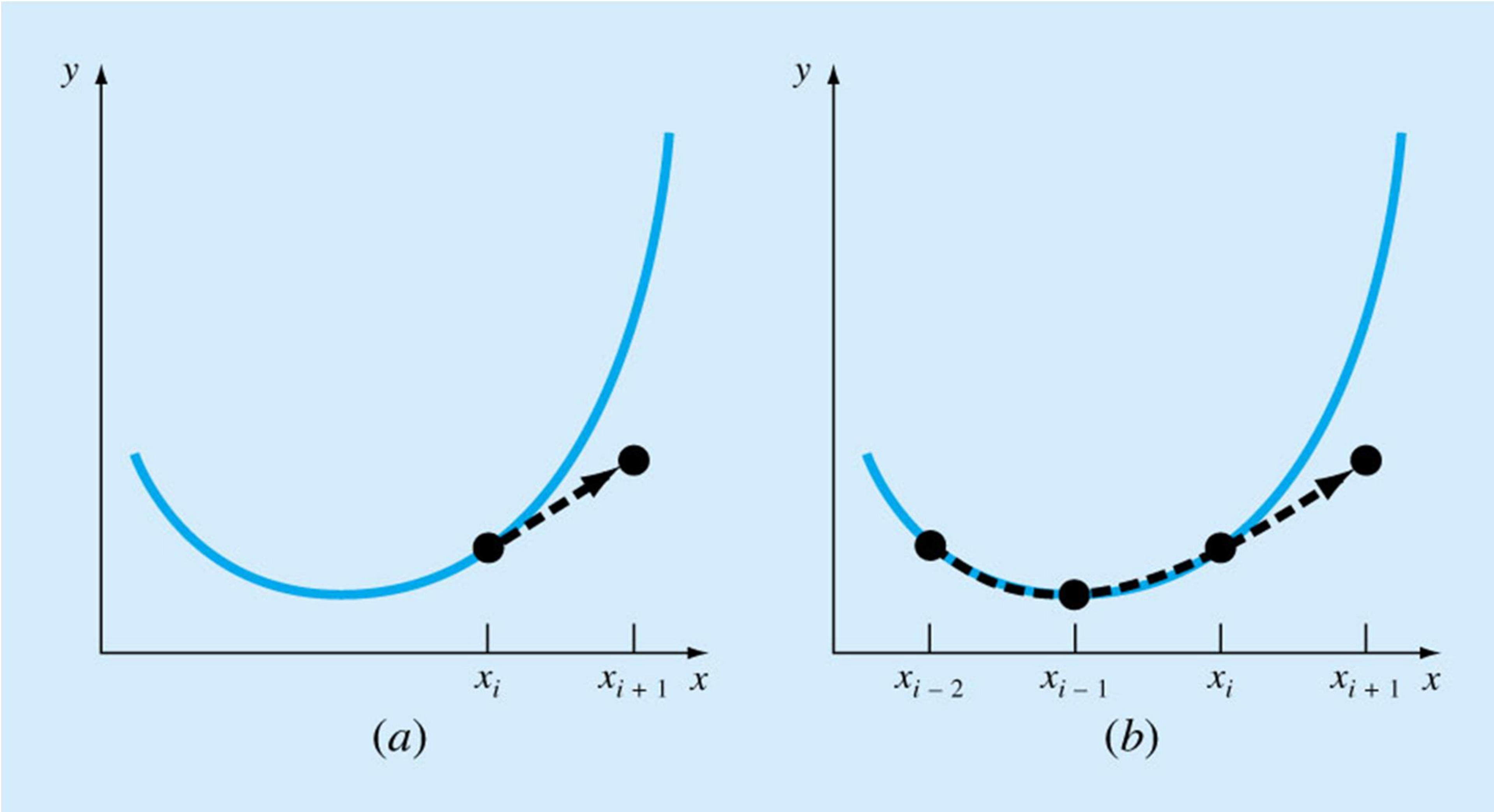
# Backward or implicit Euler's method

$$y_{i+1} = y_i + \frac{dy_{i+1}}{dt} h$$

$$y_{i+1} = y_i - a y_{i+1} h$$

$$y_{i+1} = \frac{y_i}{1 + ah}$$




y
x
$x_i$
$x_{i+1}$
(a)
y
x
$x_{i-2}$
$x_{i-1}$
$x_i$
$x_{i+1}$
(b)

# Multistep Methods

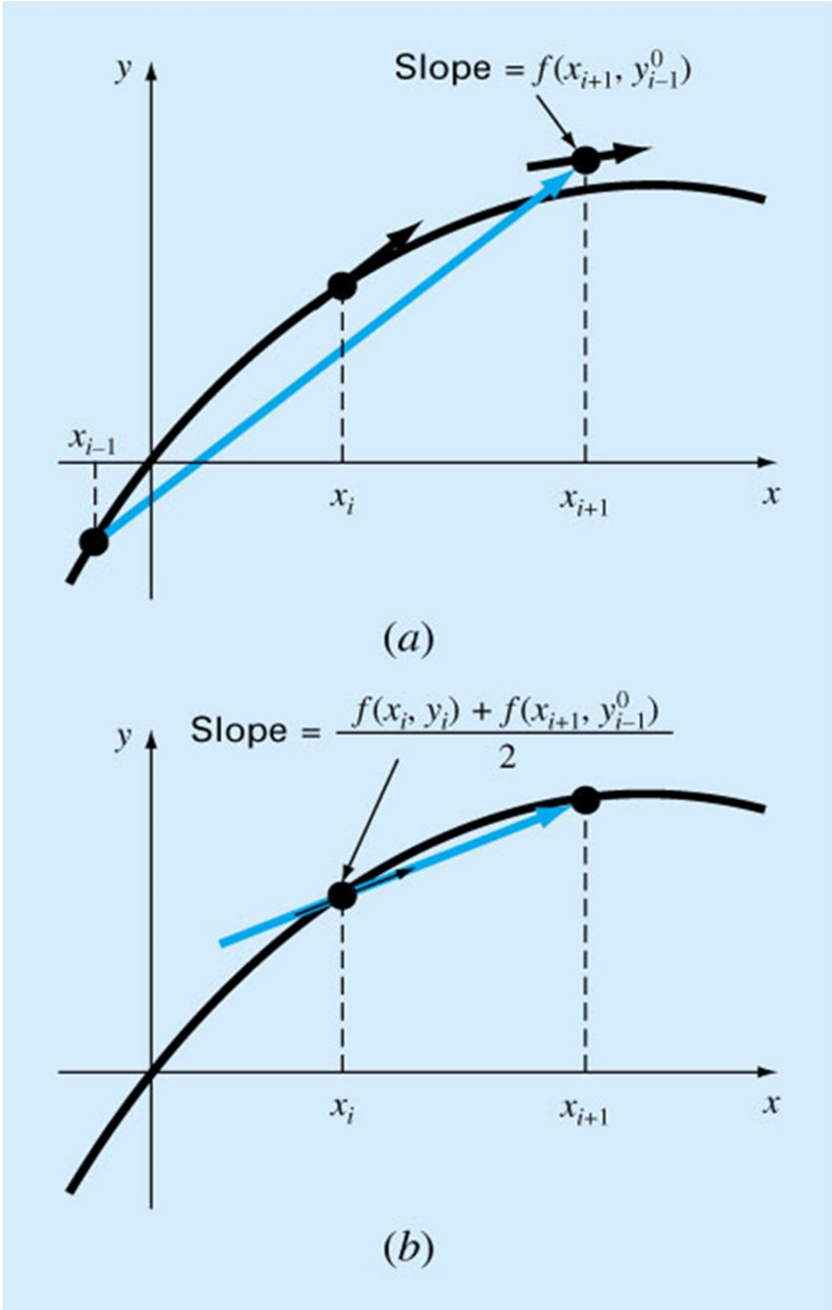
$$y_{i+1}^{0} = y_{i-1} + f(x_i, y_i)2h$$

Huen method uses Euler's method as a predictor and trapezoidal rule as a corrector.

Predictor is the weak link in the method because it has the greatest error, O(h2).

One way to improve Heun's method is to develop a predictor that has a local error of O(h3).


$$y_{i+1}^{0} = y_{i-1} + f(x_i, y_i)2h$$



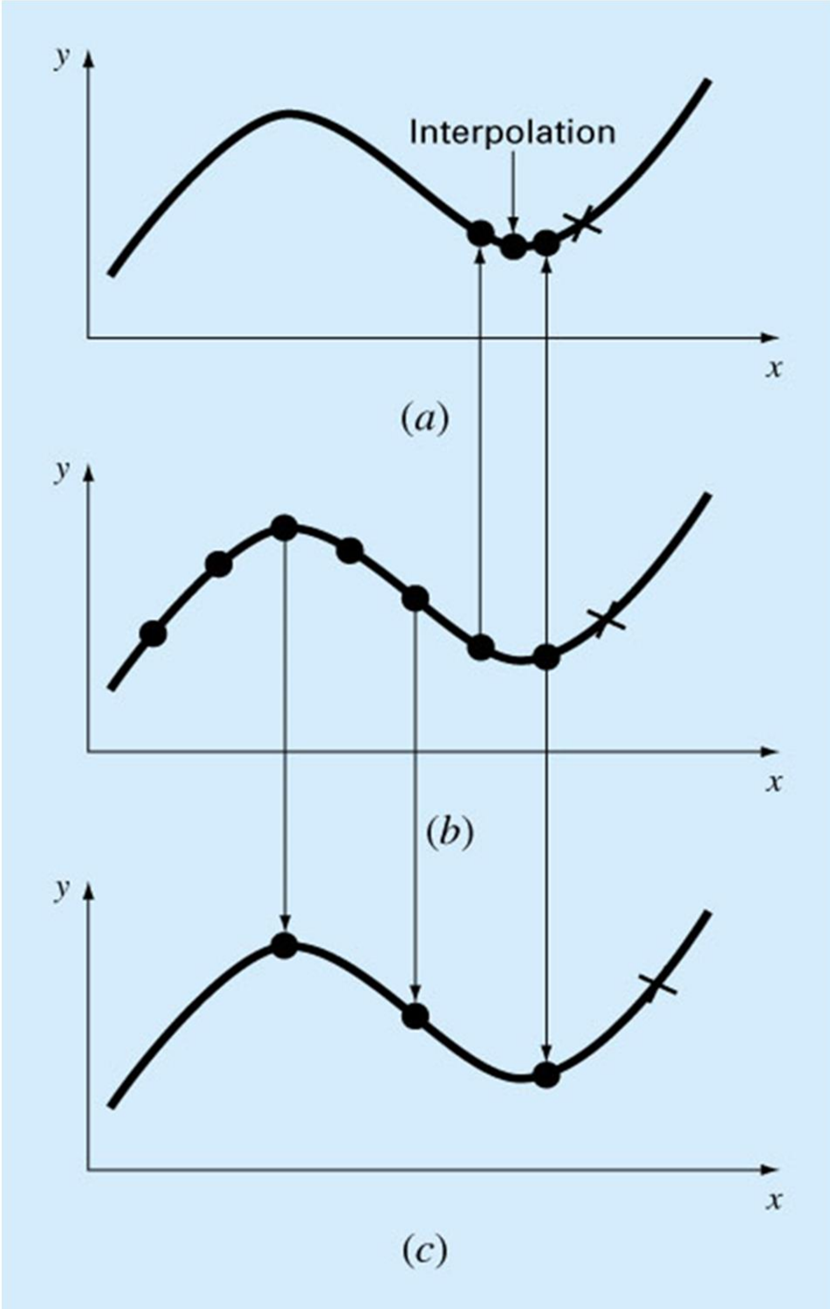
y
Slope = $f(x_{i+1}, y_{i-1}^0)$
$x_{i-1}$
$x_i$
$x_{i+1}$
x
(a)
y
Slope = $\frac{f(x_i, y_i) + f(x_{i+1}, y_{i-1}^0)}{2}$
$x_i$
$x_{i+1}$
x
(b)

Constant Step Size.

A value for h must be chosen prior to computation.
It must be small enough to yield a sufficiently small truncation error.
It should also be as large as possible to minimize run time cost and round-off error.

Variable Step Size.

If the corrector error is greater than some specified error, the step size is decreased.
A step size is chosen so that the convergence criterion of the corrector is satisfied in two iterations.
A more efficient strategy is to increase and decrease by doubling and halving the step size.


y
Interpolation
x
(a)
y
x
(b)
y
x
(c)

Integration Formulas/

Newton-Cotes Formulas.

Open Formulas.

$$y_{i+1} = y_{i-n} + \int_{x_{i-n}}^{x_{i+1}} f_n(x)dx$$

Closed Formulas.

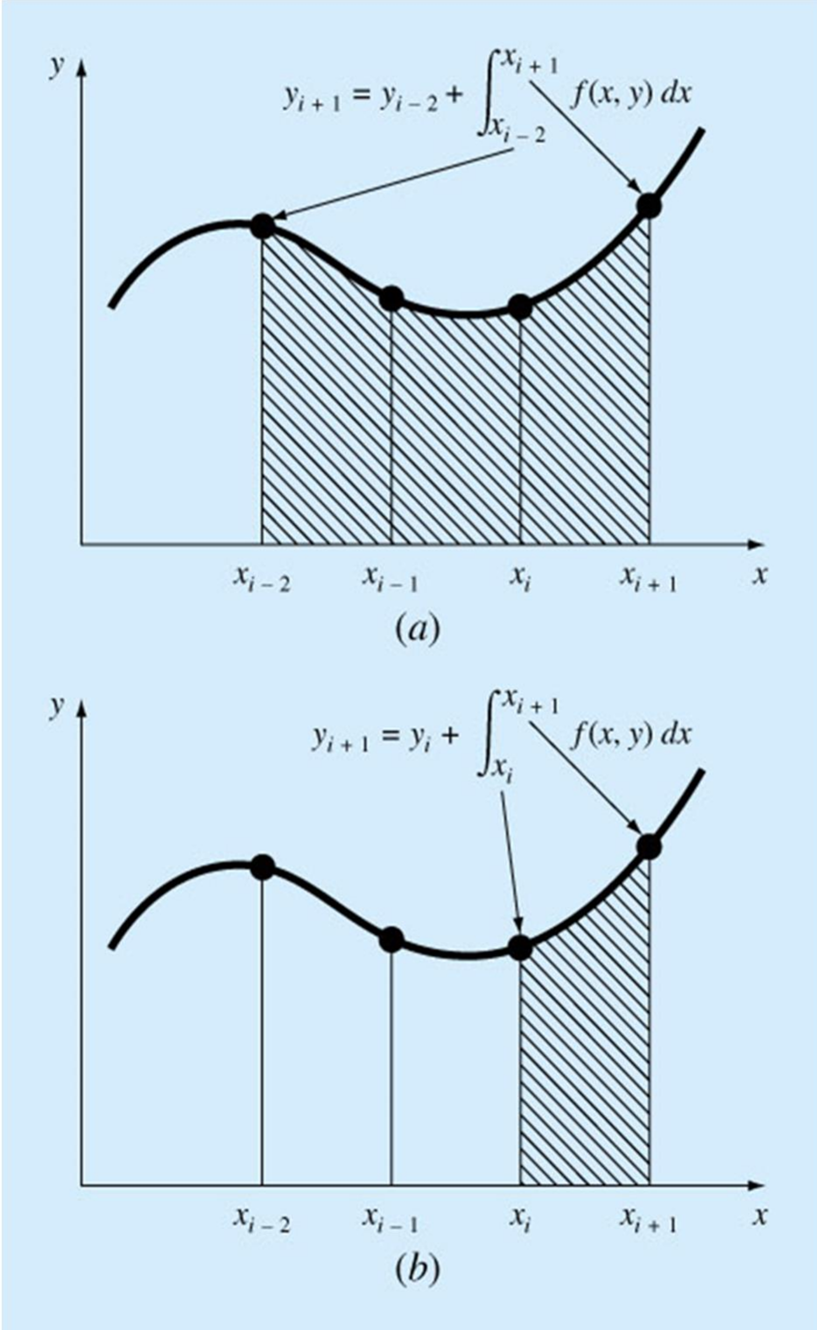
$$y_{i+1} = y_{i-n+1} + \int_{x_{i-n+1}}^{x_{i+1}} f_n(x)dx$$

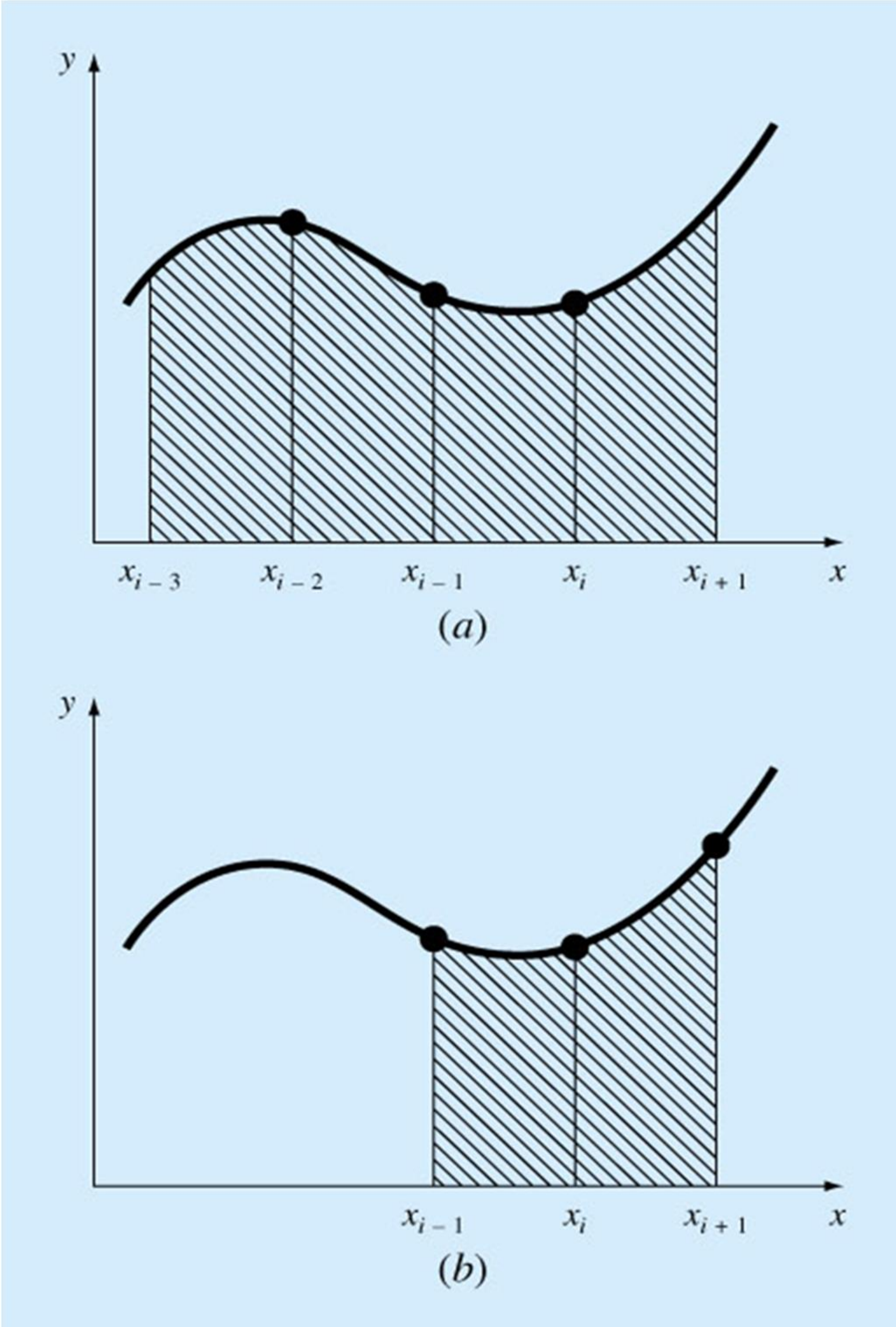
Open Formula

$$y_{i+1} = y_i + h\left(\frac{3}{2}f_i - \frac{1}{2}f_{i-1}\right) + \frac{5}{12}h^3 f_i'' + O(h^4)$$

Closed Formula

$$y_{i+1} = y_i + h\sum_{k=0}^{n-1} \beta_k f_{i+1-k} + O(h^{n+1})$$




y
x
$x_{i-3}$
$x_{i-2}$
$x_{i-1}$
$x_i$
$x_{i+1}$
(a)
y
x
$x_{i-1}$
$x_i$
$x_{i+1}$
(b)

# Higher-Order multistep Methods/

## Milne's Method.

Uses the three point Newton-Cotes open formula as a predictor and three point Newton-Cotes closed formula as a corrector.

## Fourth-Order Adams Method.

Based on the Adams integration formulas. Uses the fourth-order Adams-Bashforth formula as the predictor and fourth-order Adams-Moulton formula as the corrector.