Physics 122: Electricity &Magnetism

Chapter 24: Gauss's Law

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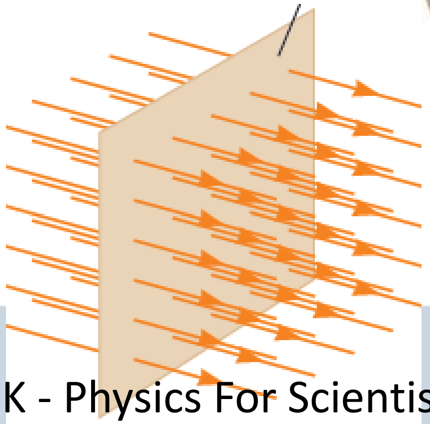
Flux of Electric Field

a slab



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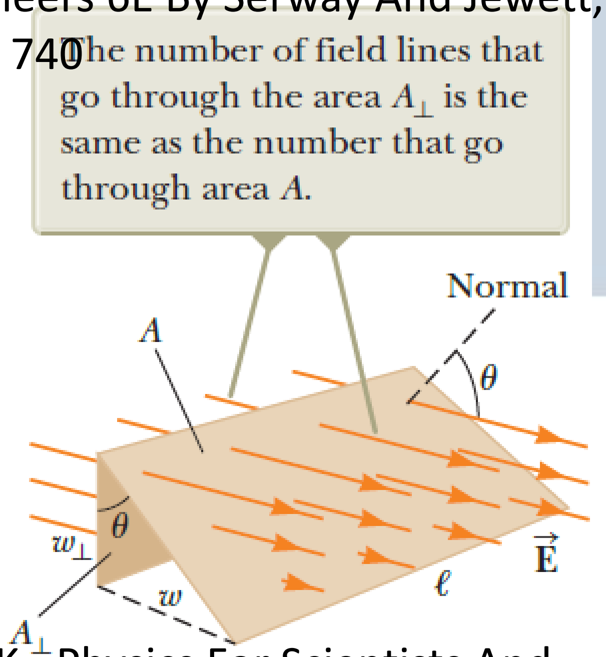
- We can think of electric field as flowing through a surface. (although actually nothing is moving)
- We represent the flux of electric field as Φ, so the flux of the electric field through an element of area ΔA is
  $\Delta\Phi = \mathbf{E}.\ \Delta\mathbf{A} = E.\ \Delta A.\cos\theta$
- When θ <90°, the flux is positive (out of the surface) and when θ>90°, the flux is negative.



$$E\cos\theta \Rightarrow Component\ of\ \vec{E}\ that\ is\ normal\ to\ the\ surface$$

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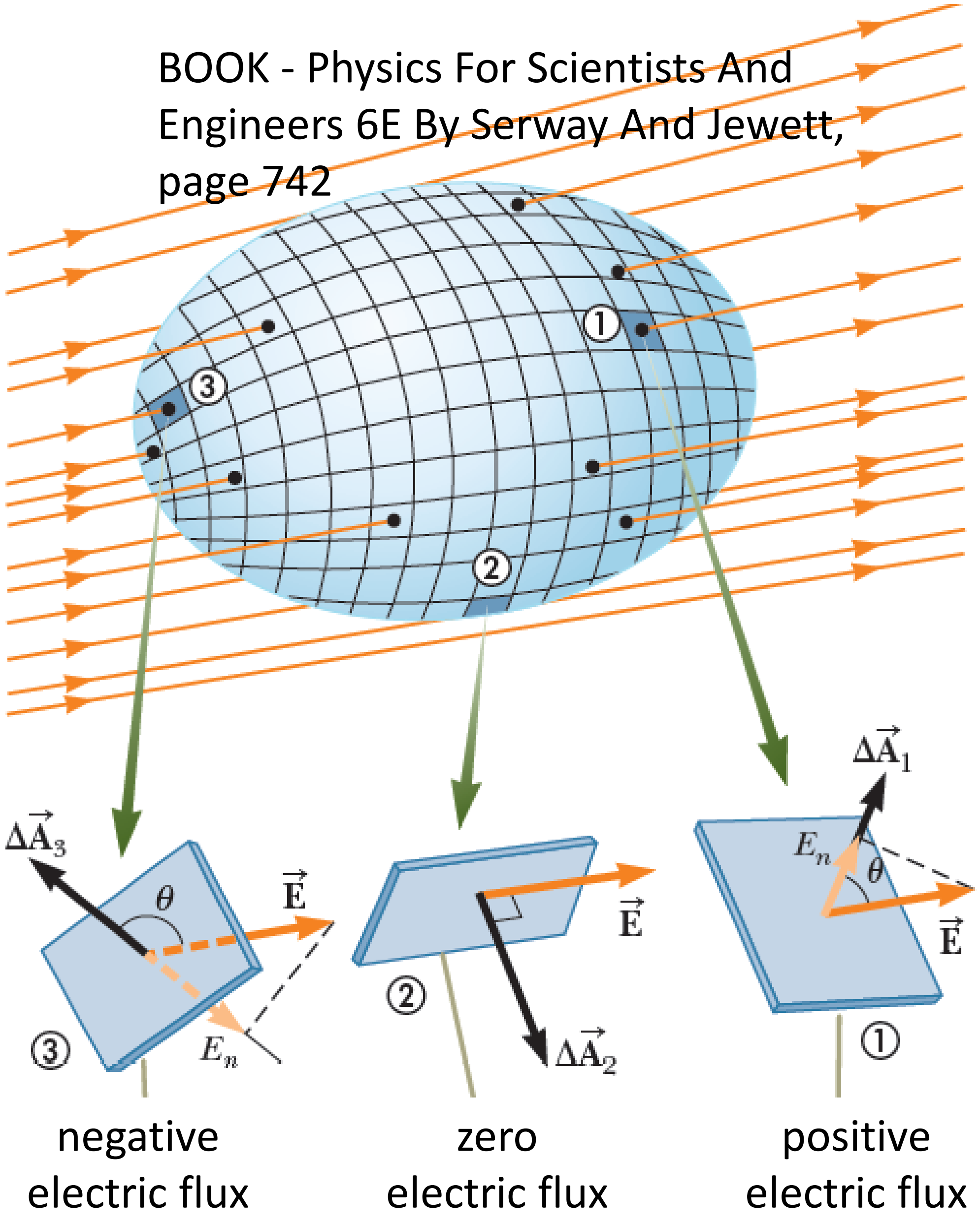


- The net flux is proportional to the number of lines leaving the surface minus the number of lines entering the surface.
- If more lines are leaving than entering, the net flux is positive.
- If more lines are entering than leaving, the net flux is negative.

We can write the net flux through a closed surface as,

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E_n \, dA$$

where E_n represents the component of the electric field normal to the surface.

Example 24.1 Flux Through a Cube

Consider a uniform electric field $\vec{\mathbf{E}}$ oriented in the x direction in empty space. A cube of edge length ℓ is placed in the field, oriented as shown in Figure 24.5. Find the net electric flux through the surface of the cube.

Solution:

The net flux can be evaluated by summing up the fluxes through each face of the cube.
Note that the flux through four of the faces is zero because $\vec{E}$ is perpendicular to d$\vec{A}$ on these faces (3, 4, 5, 6).
Consider the faces labeled 1 and 2.
The net flux through these faces is:

$$\Phi_E = \int_1 \vec{E}.d\vec{A} + \int_2 \vec{E}.d\vec{A} = -EA + EA = 0 \quad (A = l^2)$$

Hence the net flux over all of the faces is zero.

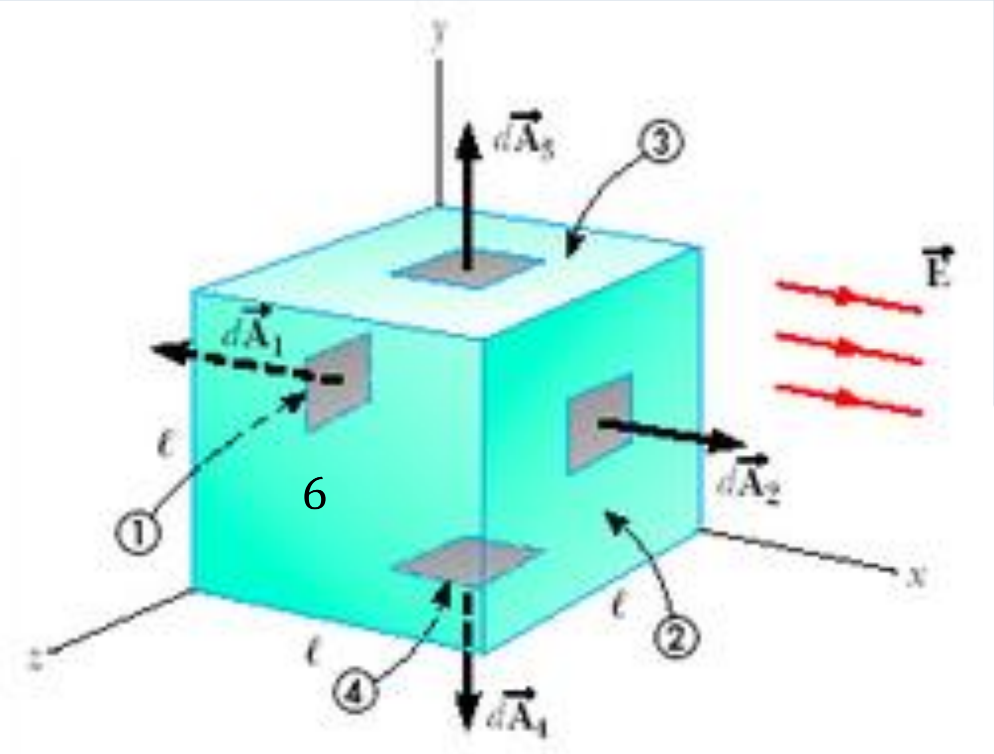


Figure 24.5

Gauss' Law

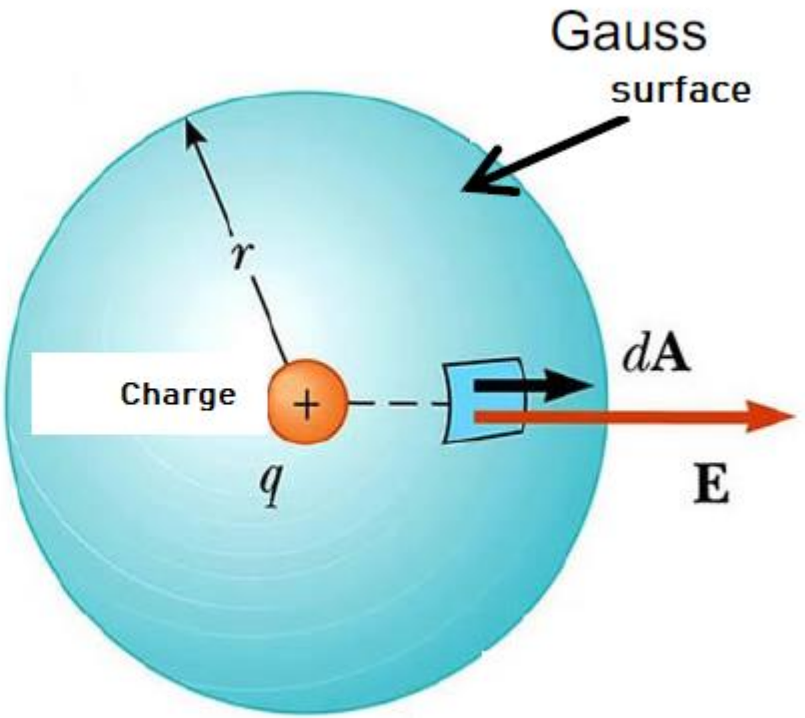
- We are going to be most interested in *closed* surfaces, in which case the outward direction becomes self-evident.
- We can ask, what is the electric flux out of such a closed surface? Just integrate over the closed surface: $\Phi = \oint d\Phi = \oint \vec{E} \cdot d\vec{A}$

Flux positive => out
Flux negative => in

- The $\oint$ symbol has a little circle to indicate that the integral is over a closed surface.
- The closed surface is called a gaussian surface, because such surfaces are used by Gauss' Law, which states that:

Gauss' Law

The flux of electric field through a closed surface is proportional to the charge enclosed.



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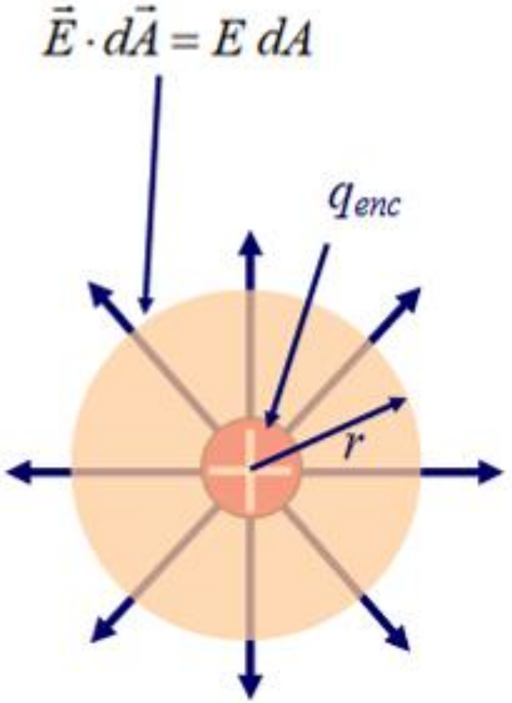
Mathematical Statement of Gauss' Law

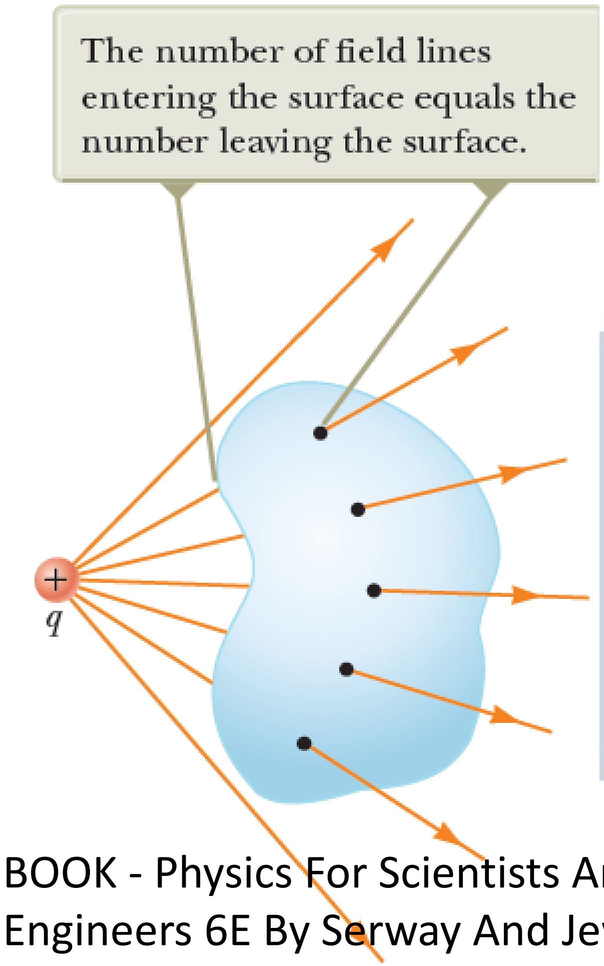
$$E = k_e q / r^2 \qquad \oint dA = A = 4\pi r^2$$

$$\Phi_E = k_e \frac{q}{r^2}(4\pi r^2) = 4\pi k_e q$$

$$k_e = 1/4\pi\epsilon_0 \longrightarrow \Phi_E = \frac{q}{\epsilon_0}$$



The net flux through the spherical surface is proportional to the charge inside the surface.
The flux is independent of the radius r.



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The net electric flux through a closed surface that surrounds no charge is zero.

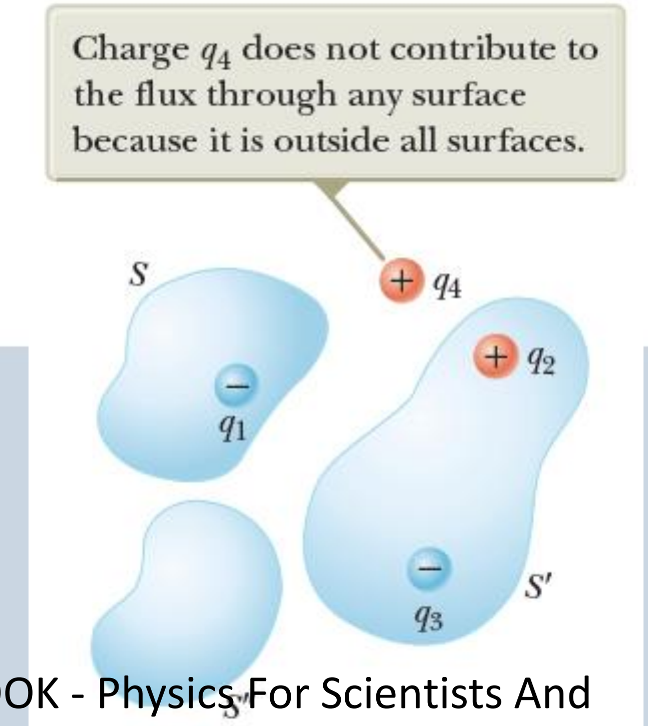

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Figure 24.9 The net electric flux through any closed surface depends only on the charge *inside* that surface. The net flux through surface S is q_1/ϵ_0, the net flux through surface S' is $(q_2 + q_3)/\epsilon_0$, and the net flux through surface S'' is zero.

Gauss's law states that the net flux through any closed surface is,

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{in}}}{\epsilon_0}$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \; SI \; units$$

where $\vec{E}$ represents the electric field at any point on the surface and q_{in} represents the net charge inside the surface. Note that $\vec{E}$ represents the total electric field, which includes contributions from charges both inside and outside the surface.

Gauss's law is applicable only in a limited number of **highly symmetric** situations.

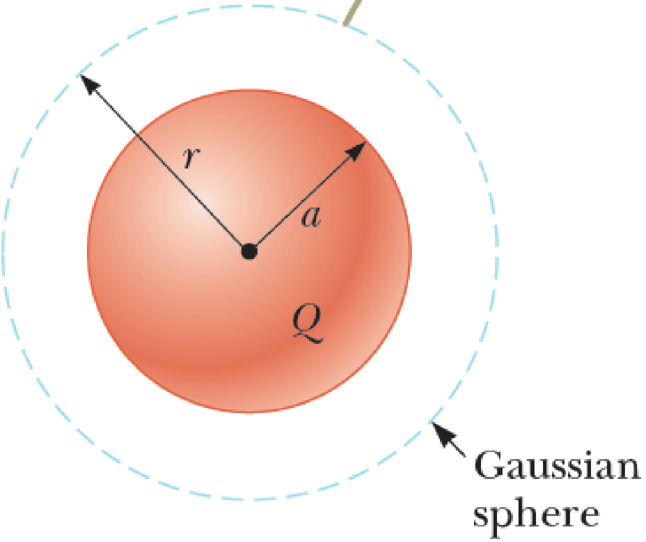
In the next section we use Gauss's law to evaluate the electric field for charge distributions that have spherical, cylindrical, or planar symmetry. If one chooses the gaussian surface surrounding the charge distribution carefully, the integral in can be simplified. We should always take advantage of the symmetry of the charge distribution.

Example 24.3 A Spherically Symmetric Charge Distribution

An insulating solid sphere of radius *a* has a uniform volume charge density ρ and carries a total positive charge *Q*.

(A) Calculate the magnitude of the electric field at a point outside the sphere.

For points outside the sphere, a large, spherical gaussian surface is drawn concentric with the sphere.



$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\, dA = \frac{Q}{\epsilon_0}$$

$$\oint E\, dA = E \oint dA = E(4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2} = \boxed{k_e \frac{Q}{r^2} \quad (\text{for } r > a)}$$

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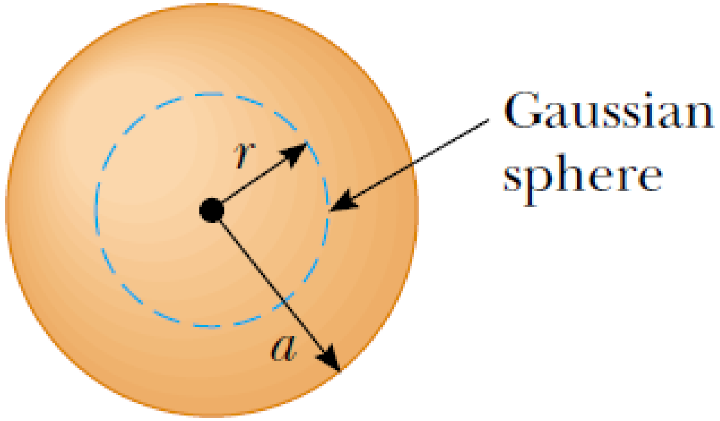
(B) Find the magnitude of the electric field at a point inside the sphere.

Solution: In this case we select a spherical gaussian surface having radius *r* <*a*

$$q_{\text{in}} = \rho V' = \rho\left(\tfrac{4}{3}\pi r^3\right)$$



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By symmetry the magnitude of the electric field is constant everywhere on the spherical gaussian surface and is normal to the surface at each point.

$$\oint E\,dA = E\oint dA = E\,(4\pi r^2) = \frac{q_{\text{in}}}{\epsilon_0}$$

$$E = \frac{q_{\text{in}}}{4\pi\epsilon_0 r^2} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0}\,r$$

$\rho = Q/\frac{4}{3}\pi a^3$ and $k_e = 1/4\pi\epsilon_0$ $\Longrightarrow$ $E = \dfrac{Qr}{4\pi\epsilon_0 a^3} = \boxed{k_e \dfrac{Q}{a^3}\, r}$ $\quad$ $(\text{for } r < a)$

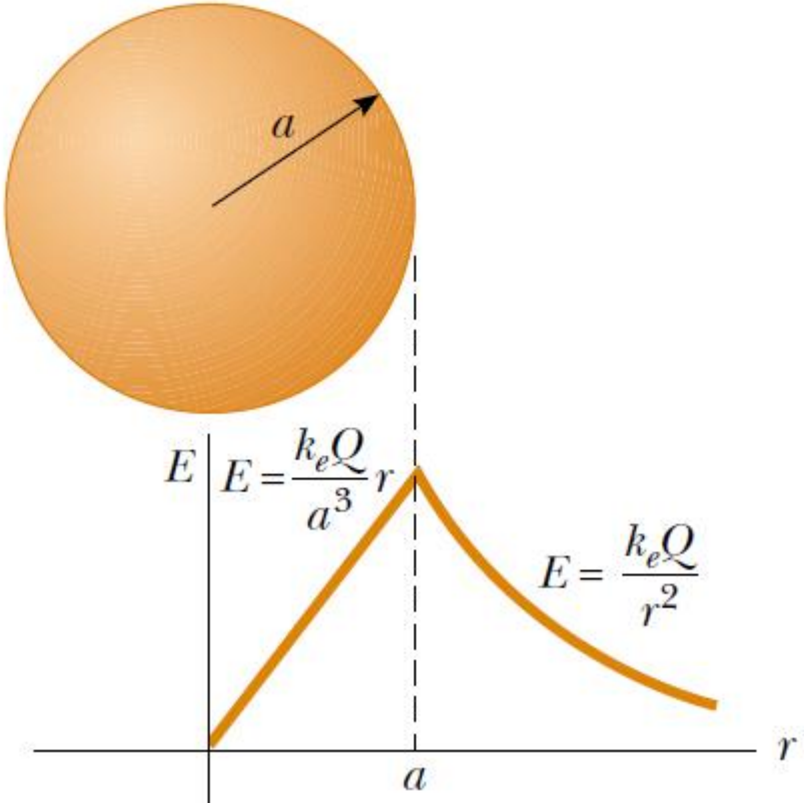
It shows that *E* -> 0 as *r* -> 0.

What If? Suppose we approach the radial position $r = a$ from inside the sphere and from outside. Do we measure the same value of the electric field from both directions?

Solution:

$$E = \lim_{r \to a} \left(k_e \frac{Q}{r^2} \right) = k_e \frac{Q}{a^2}$$ from outside

$$E = \lim_{r \to a} \left(k_e \frac{Q}{a^3} r \right) = k_e \frac{Q}{a^3} a = k_e \frac{Q}{a^2}$$ from inside



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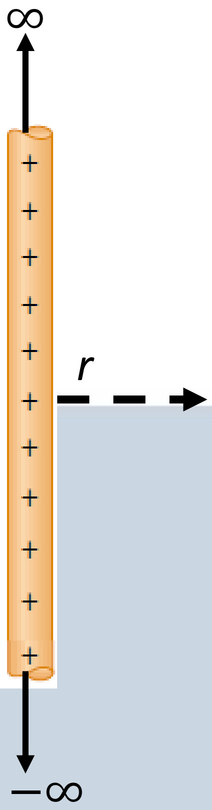
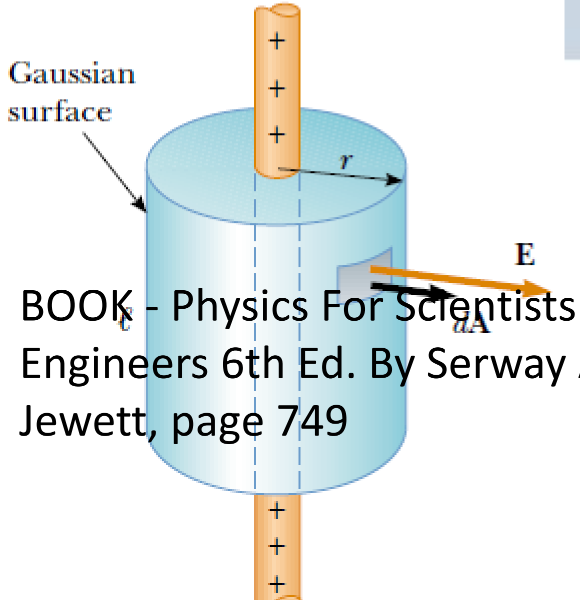
Example 24.4 A Cylindrically Symmetric Charge Distribution

Find the electric field a distance *r* from a line of positive charge of infinite length and constant charge per unit length λ.



Solution: The line of charge is infinitely long. Therefore, the field is the same at all points equidistant from the line. The charge distribution has cylindrical symmetry and we can apply Gauss's law to find the electric field. Lets choose a cylindrical gaussian surface of radius *r* and length l that is coaxial with the line charge. Because E.*d*A=0 for the ends of the cylinder, we restrict our attention to only the curved surface of the cylinder.



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$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = E \oint dA = EA = \frac{q_{in}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0}$$

$$E(2\pi r \ell) = \frac{\lambda \ell}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = 2k_e \frac{\lambda}{r}$$

Note that a finite line charge does not possess sufficient symmetry for us to make use of Gauss's law.

Electric field due to a cylindrically symmetric charge distribution varies as 1/r.

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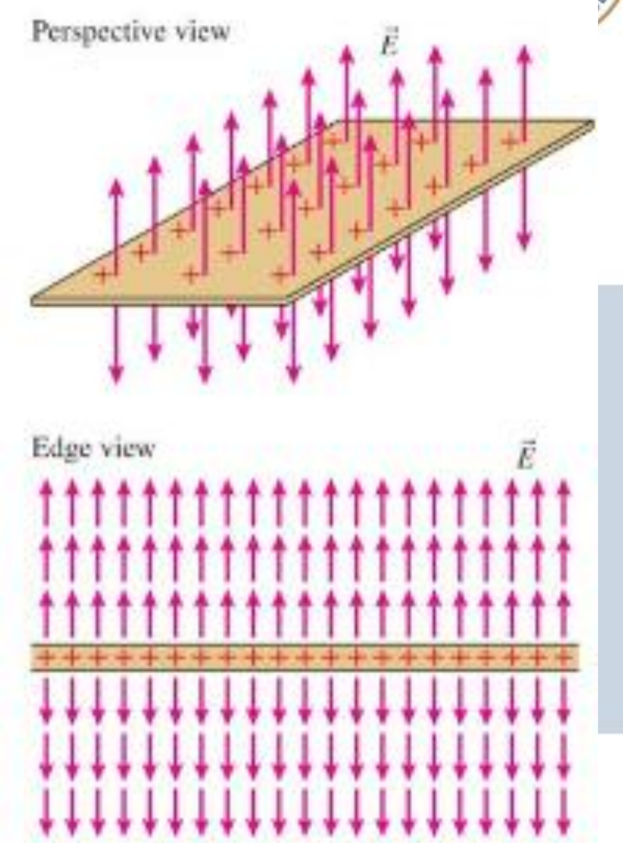
Example 24.5 A Plane of Charge

Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.



Solution: ***E*** must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane.

A gaussian surface that reflects the symmetry is a small cylinder whose axis is perpendicular to the plane. There is no contribution to the surface integral from the curved surface (because ***E*** is parallel)

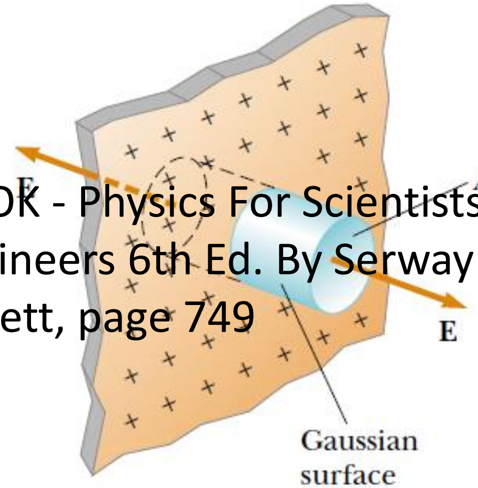


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Figure 24.15 (Example 24.8) A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is EA through each end of the gaussian surface and zero through its curved surface.

$\Phi_E = 2EA$

$q_{in} = \sigma A$

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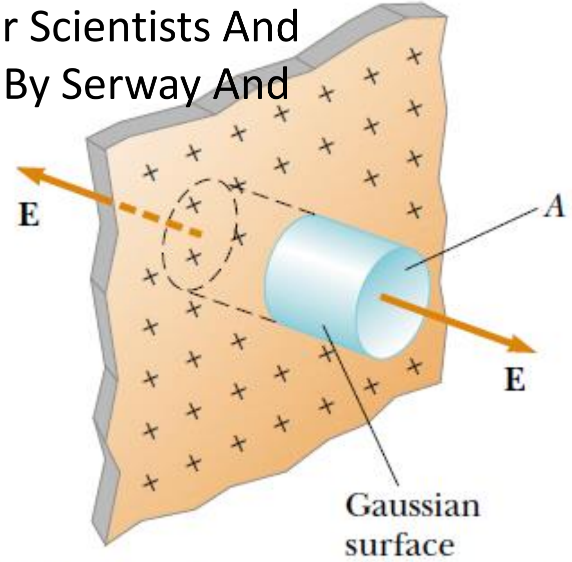

Figure 24.15 (Example 24.8) A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is EA through each end of the gaussian surface and zero through its curved surface.

$$\Phi_E = 2EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

Because the distance from each flat end of the cylinder to the plane does not appear in the ***E***, we conclude that $E=\frac{\sigma}{\varepsilon_0}$ at *any* distance from the plane.

The electric field between two infinite planes of charge parallel to each other, one positively charged and the other negatively charged, results in a uniform field $\sigma/\ \varepsilon_0$ in the region between the planes and cancel elsewhere to give a field of zero.

CONDUCTORS

FIELDS INSIDE A CONDUCTOR

When there is no net motion of charge within a conductor, the conductor is in electrostatic equilibrium.
A conductor in electrostatic equilibrium has the following properties:

1. The electric field is zero everywhere inside the conductor (whether solid or hollow).
2. If an isolated conductor carries a charge, the charge resides on its surface.
3. The electric field just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude σ/ε_0.
4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.

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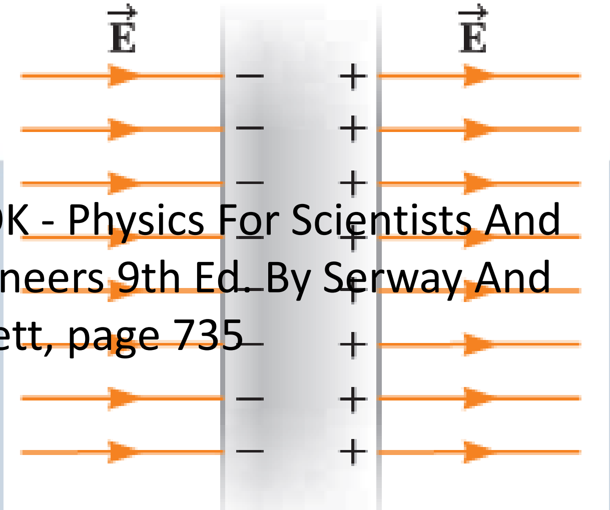

Figure 24.16 A conducting slab in an external electric field $\vec{E}$. The charges induced on the two surfaces of the slab produce an electric field that opposes the external field, giving a resultant field of zero inside the slab.

Example 24.7 A Sphere Inside a Spherical Shell

A solid conducting sphere of radius *a* carries a net positive charge 2*Q* uniformly distributed throughout its volume. A conducting spherical shell of inner radius *b* and outer radius *c* is concentric with the solid sphere and carries a net charge -*Q*. Using Gauss's law, find the electric field in the regions labeled 1, 2, 3, and 4 and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

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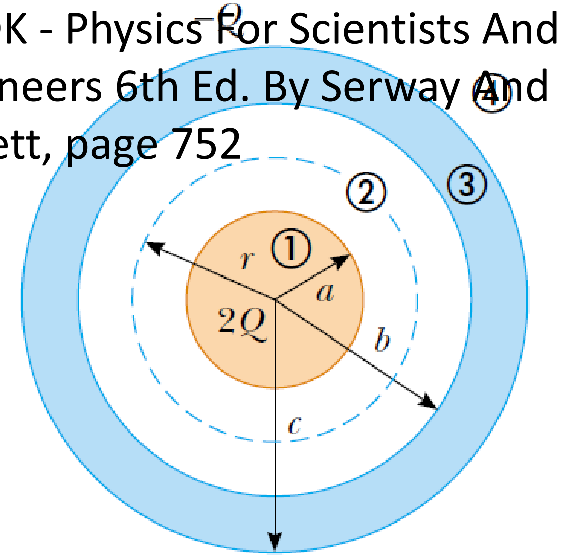

Figure 24.20 (Example 24.10) A solid conducting sphere of radius a and carrying a charge $2Q$ surrounded by a conducting spherical shell carrying a charge $-Q$.

Solution: To determine the electric field at various distances *r* from this center, we construct a spherical gaussian surface for each of the four regions.

Region 1: $q_{\text{in}} = 0 \Rightarrow E_1 = 0 \text{ for } r < a$

Region 2: $E_2 A = E_2(4\pi r^2) = \dfrac{q_{\text{in}}}{\epsilon_0} = \dfrac{2Q}{\epsilon_0}$

$$E_2 = \frac{2Q}{4\pi\epsilon_0 r^2} = \boxed{\frac{2k_eQ}{r^2}} \qquad (\text{for } a < r < b)$$

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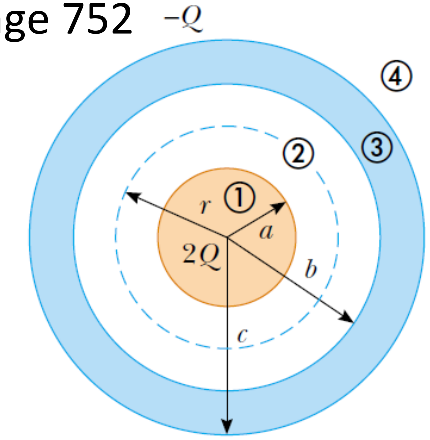


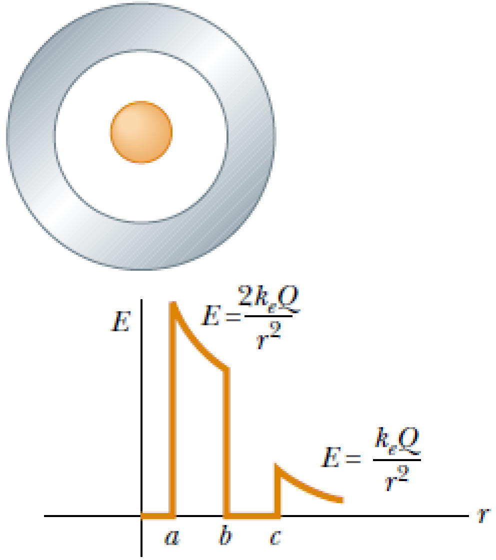
Figure 24.20 (Example 24.10) A solid conducting sphere of radius a and carrying a charge $2Q$ surrounded by a conducting spherical shell carrying a charge $-Q$.

Region 3: The electric field must be zero because the spherical shell is also a conductor in equilibrium. $q_{in}=0$.

Region 4: $q_{\text{in}} = 2Q + (-Q) = Q$

$$E_4 = \frac{k_e Q}{r^2} \qquad (\text{for } r > c)$$

The charge on the inner surface of the spherical shell must be $-2Q$ to cancel the charge $+2Q$ on the solid sphere. Because the net charge on the shell is $-Q$, we conclude that its outer surface must carry a charge $+Q$.



Summary

- Electric flux is proportional to the number of electric field lines that penetrate a surface and is generally defined as:

$$\Phi_E \equiv \int_{\text{surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

- Gauss's law says that the net electric flux Φ_E through any closed gaussian surface is equal to the *net* charge q_{in} inside the surface divided by ε_0.

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{in}}}{\epsilon_0}$$

- Insulating sphere, line and plane charge:

E field due to a	falls off at
point charge	$1/r^2$
line of charge	$1/r^1$
plane of charge	$1/r^0$

- The electric field is zero everywhere inside the conductor and any net charge on the conductor resides entirely on its surface. The electric field just outside the conductor is σ/ε_0.