Calculus
Lecture 2

Oktay Olmez and Serhan Varma

Limit: Intuitive Meaning

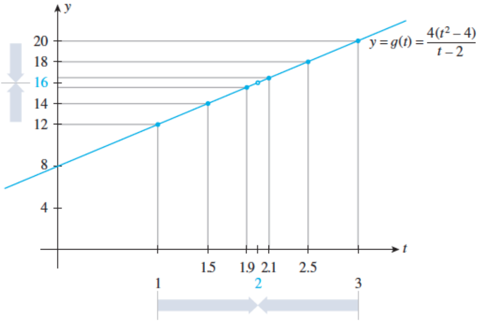
Definition

Let f be a real valued function. To say that $\lim_{x \to c} f(x) = L$ means that when x is near but different from c then $f(x)$ is near L.

Limit: Intuitive Meaning

Definition

Let f be a real valued function. To say that $\lim_{x \to c} f(x) = L$ means that when x is near but different from c then $f(x)$ is near L.



Limit: Mathematical Definition

Definition

Let $f : D \to \mathbb{R}$ *be a function defined on a subset* $D \subseteq \mathbb{R}$ *, let* c *be a limit point of* D*, and let* L *be a real number. Then the function* f *has a limit* L *at* c *is defined to mean for all* $\varepsilon > 0$ *, there exists a* $\delta > 0$ *such that for all* x *in* D *that satisfy* $0 < |x - c| < \delta$ *, the inequality* $|f(x) - L| < \varepsilon$ *holds.*

Limit: Mathematical Definition

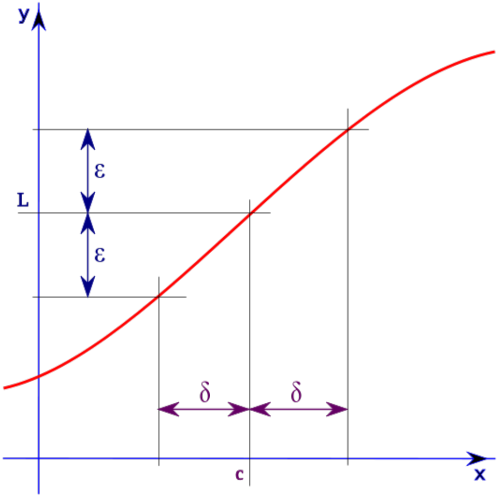
Definition

Let $f : D \to \mathbb{R}$ be a function defined on a subset $D \subseteq \mathbb{R}$, let c be a limit point of D, and let L be a real number. Then the function f has a limit L at c is defined to mean for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x in D that satisfy $0 < |x - c| < \delta$, the inequality $|f(x) - L| < \varepsilon$ holds.

Symbolically:

$$\lim_{x \to c} f(x) = L \iff (\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall x \in D)(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon).$$

Limit: Mathematical Definition



Limit: Mathematical Definition

Question

Prove the statement that

$$\lim_{x \to 5} (3x - 3) = 12.$$

Limit: Mathematical Definition

Question

Prove the statement that

$$\lim_{x \to 5}(3x - 3) = 12.$$

Solution

According to the formal definition, the statement is true if and only if confining x *to* δ *units of* 5 *will inevitably confine* $3x - 3$ *to* ε *units of* 12*. The overall key to showing this implication is to demonstrate how* δ *and* ε *must be related to each other such that the implication holds. Mathematically, we want to show that*

$$0 < |x - 5| < \delta \;\Rightarrow\; |(3x - 3) - 12| < \varepsilon.$$

Simplifying the equation yields $|x - 5| < \varepsilon/3$*. Thus,* δ *and* ε *is related by* $\delta \leq \varepsilon/3$.

One-sided Limits

$\lim_{x \to c^+} f(x)$ means that x approaches c from right.

One-sided Limits

$\lim_{x \to c^+} f(x)$ means that x approaches c from right.
$\lim_{x \to c^-} f(x)$ means that x approaches c from left.

One-sided Limits

$\lim\limits_{x \to c^+} f(x)$ means that x approaches c from right.
$\lim\limits_{x \to c^-} f(x)$ means that x approaches c from left.

Definition

Let f be a real valued function. To say that $\lim\limits_{x \to c^+} f(x) = L$ means that when x is near but to the right of c then $f(x)$ is near L.

One-sided Limits

$\lim_{x \to c^+} f(x)$ means that x approaches c from right.
$\lim_{x \to c^-} f(x)$ means that x approaches c from left.

Definition

Let f be a real valued function. To say that $\lim_{x \to c^+} f(x) = L$ means that when x is near but to the right of c then $f(x)$ is near L.

Definition

Let f be a real valued function. To say that $\lim_{x \to c^-} f(x) = L$ means that when x is near but to the left of c then $f(x)$ is near L.

One-sided Limits

$\lim_{x \to c^+} f(x)$ means that x approaches c from right.
$\lim_{x \to c^-} f(x)$ means that x approaches c from left.

Definition

Let f be a real valued function. To say that $\lim_{x \to c^+} f(x) = L$ means that when x is near but to the right of c then $f(x)$ is near L.

Definition

Let f be a real valued function. To say that $\lim_{x \to c^-} f(x) = L$ means that when x is near but to the left of c then $f(x)$ is near L.

Theorem

$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$ *if and only if* $\lim_{x \to c} f(x) = L$

Heaviside Function: The Heaviside function is often used to specify when something is on or off

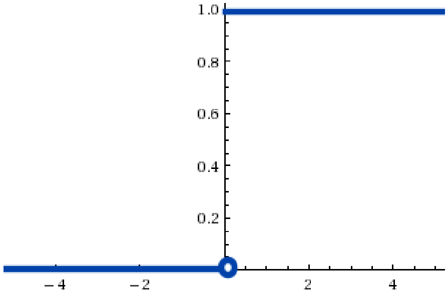
The Heaviside function is defined as

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Heaviside Function: The Heaviside function is often used to specify when something is on or off

The Heaviside function is defined as

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$



Heaviside Function

- This function clearly has the limit of 0 for any $x < 0$, and it has the limit of 1 for any $x > 0$.

Heaviside Function

- This function clearly has the limit of 0 for any $x < 0$, and it has the limit of 1 for any $x > 0$.
- Even though this function is defined to be 1 at $x = 0$, it does not have a limit at $x = 0$

An Application: Free-Fall

Example

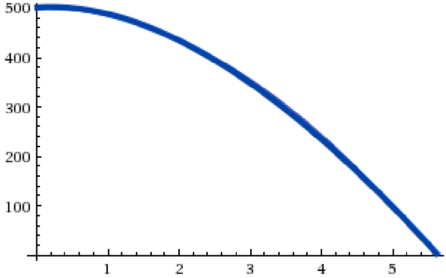
Use the position function $s(t) = -16t^2 + 500$ *which gives the height (in feet) of an object that has fallen for seconds from a height of* 500 *feet. The velocity at time* $t = a$ *seconds is given by*

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}$$

(a) *If a construction worker drops a wrench from a height of 500 feet, how fast will the wrench be falling after 2 seconds?*

(b) *If a construction worker drops a wrench from a height of 500 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground?*

Solution



(a) $\displaystyle\lim_{t\to 2} \frac{s(2)-s(t)}{2-t} = -64$

(b) $\displaystyle\lim_{t\to \frac{5\sqrt{5}}{2}} \frac{s(\frac{5\sqrt{5}}{2})-s(t)}{\frac{5\sqrt{5}}{2}-t} = -80\sqrt{5}$