Calculus
Lecture 3

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Continuity at a point

Definition

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In other words,

- f has to be defined at c.
- $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c^+} f(x)$ exist and equal.
- The value of the limit must equal $f(c)$.

Examples of Continuous Functions

Example

A polynomial function is continuous at every real number.

Examples of Continuous Functions

Example

A polynomial function is continuous at every real number.
A rational function is continuous everywhere except where its denominator is zero.

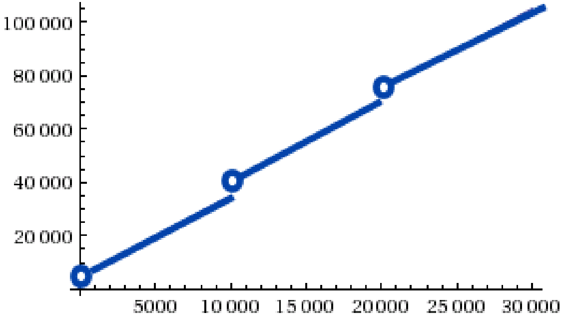
Examples of a discontinuous Functions

Example

A bookbinding company produces $10,000$ books in an eight-hour shift. The fixed cost per shift amounts to \$5000, and the unit cost per book is \$3. Using the greatest integer function, you can write the cost of producing x books as

$$C(x) = 5000\left(1 + \left\lfloor \frac{x-1}{10000} \right\rfloor\right) + 3x.$$

Solution



100 000
80 000
60 000
40 000
20 000
5000
10 000
15 000
20 000
25 000
30 000

Mathematical Definition

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Continuity of $f : I \longrightarrow \mathbb{R}$ *at* $c \in I$ *means that for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $x \in I$*:*

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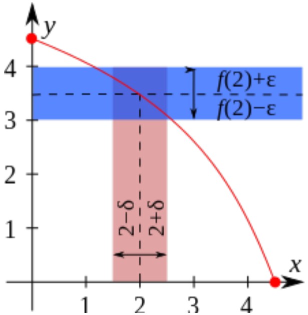

Example

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Prove that $f(x) = 2x$ at $x = 1$ is continuous.

Solution

For all $\varepsilon > 0$*, we want to find at least one associated* δ*.*

$$
\begin{aligned}
|x-1| < \delta \implies |f(x) - f(1)| &= |2x - 2| \\
&= 2|x-1| < 2\delta \leq \varepsilon \\
&\implies \delta \leq \frac{\varepsilon}{2}
\end{aligned}
$$

Continuity theorem for operations

Theorem

Let f and g be continuous at c, then so are kf, $f \pm g$, $f \cdot g$, f/g (provided that $g(c) \neq 0$), f^n and $\sqrt{f}$ (provided that $f(c) > 0$).

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Example

At what numbers is $f(x) = \dfrac{|x| - x^2}{\sqrt{x} + \sqrt[3]{x}}$ continuous?

Continuity of Trigonometric Functions

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- $\sin(x)$ and $\cos(x)$ are continuous at every real number.
- $\tan(x)$, $\cot(x)$, $\csc(x)$ and $\sec(x)$ are continuous at every real number c in their domains.

Composite Limit Theorem

Theorem

Let $\lim_{x \to c} g(x) = L$ *and* f *be a continuous function at* L*. Then,*

$$\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x)) = f(L).$$

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Example

At what numbers is $f(x) = \sin(\dfrac{x^4 - 3x + 1}{x^2 - x - 6})$ *discontinuous?*